On memory and regimes in economic time series

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Purpose and summary

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- Challenge: to understand "price" dynamics over various scales and how they may change with time.
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Summary:

- Price data have a rich multiscale/memory structure that may vary over time.
- The monitoring of these variations shows regime switches.
- The quantitative analysis is carried out by a wavelet decomposition method.

History

Fractional/memory processes:

- Kolmogorov (1940): turbulence
- Hurst (1951): hydrology $\rightarrow$ Hurst parameter H.
- Mandelbrot and Van Ness (1968): finance
- Comte and Renault (1998): fractional stochastic volatility
- Gatheral et al (2014): rough fractional stochastic volatility
- A lot of work (in mathematics, physics, biology, ...) on transport, diffusion, wave propagation in fractional media.

Fractional Brownian motion (fBM)

- Fractional Brownian motion with Hurst exponent $H \in (0,1)$:

$$B_H(t)$$

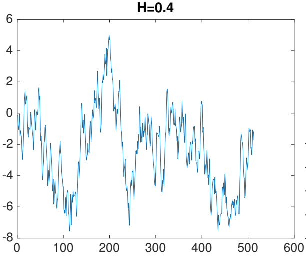
- Self-similar Gaussian process:

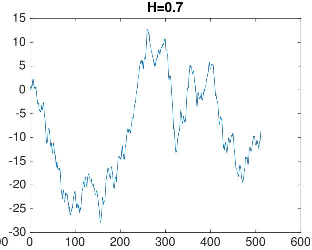
$$B_H(at) \overset{\text{probability distribution}}{\sim} a^H B_H(t) \quad \text{for any } a > 0$$

- Stationary increments:

$H = 1/2$: Independent increments (standard Brownian motion).
$H < 1/2$: Negatively correlated increments. Sample paths are continuous but rougher than BM.
$H > 1/2$: Positively correlated increments. Sample paths are not differentiable but smoother than BM.

Example Brownian Motion Paths





Standard price model

1960s: Paul A. Samuelson:

Increments of Brownian motion model "relative" price changes, essentially:

$$\frac{p(t_{n+1}) - p(t_n)}{p(t_n)} = \sigma(B(t_{n+1}) - B(t_n)),$$

with, in addition, possibly a deterministic drift.

Modeling of prices: An alternative approach

1960s: Benoit Mandelbrot:

Increments of fractional Brownian motion model "relative" price changes, essentially:

$$R_{n+1} = \frac{p(t_{n+1}) - p(t_n)}{p(t_n)} = \sigma(B_H(t_{n+1}) - B_H(t_n)).$$

$\rightarrow$ Returns have "memory":

$$\rho_H = \frac{\mathbb{E}[R_{n+1}R_n]}{\mathbb{E}[R_n^2]} = 2^{2H-1} - 1.$$

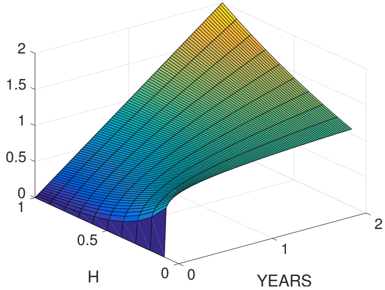
In general there is "long-range" memory:

$$\begin{aligned} \mathbb{E}[R_{n+1} \mid R_n] &= \rho_H R_n \\ &\neq \mathbb{E}[R_{n+1} \mid R_n, R_{n-1}]. \end{aligned}$$

Radical change of perspective: Hurst coefficient, H, and volatility, σ, are the fundamental quantities.

The process $\sigma B_H(t)$ scales as $\sigma|t|^H$.

Illustration: St.dev($\sigma B_H(t)$), $t \in (0,2), \sigma = 1$:



Classic case:

$H = 1/2$: independent increments.

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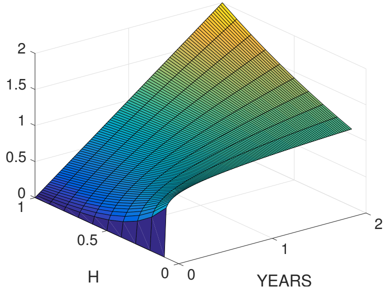

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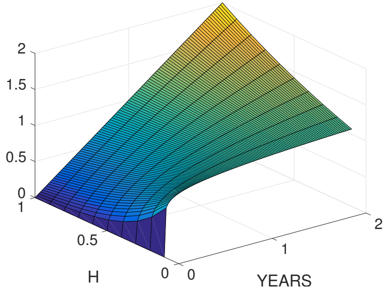


Classic case:

$H = 1/2$: independent increments.

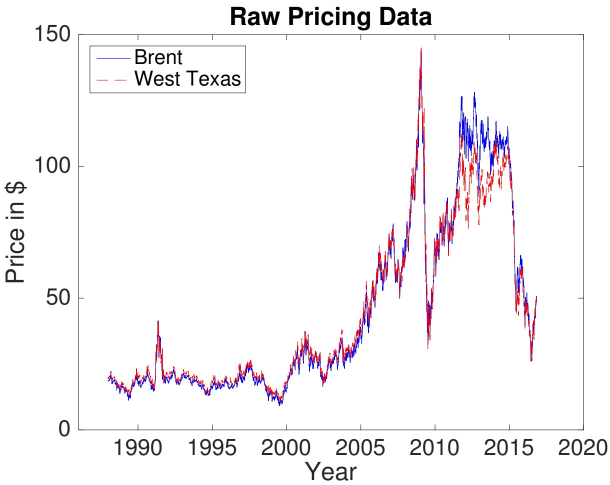
Limit cases:

$H \to 1$: Increments "equal".

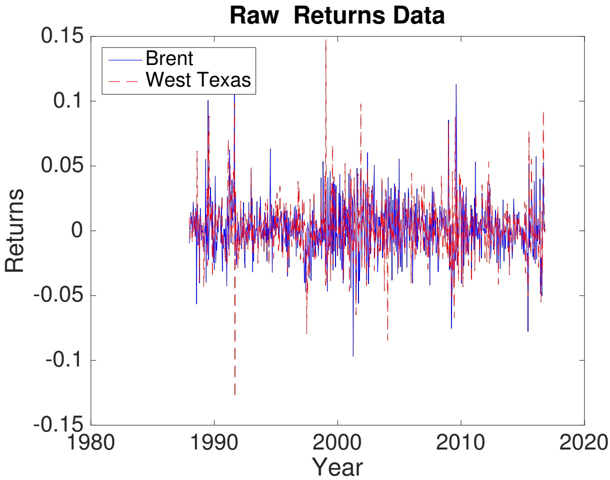
$H \to 0$: Increments "alternate in sign".

Remark: Rough case "almost" like a two-factor short- and long-scale model.

Crude Prices



Oil price data from 1987 to 2017 for West Texas (red dashed line) and Brent (solid blue line).



Returns for West Texas Crude (red dashed line) and Brent Crude (solid blue line).

$$R_{n+1} = \frac{p(t_{n+1}) - p(t_n)}{p(t_n)}, \quad t_n = n\Delta t$$

Multi-fractional price model

$\rightarrow$ Motivated by the data, let increments of multi-fractional Brownian motion model "relative" price changes, essentially:

$$R_{n+1} = \frac{p(t_{n+1}) - p(t_n)}{p(t_n)} = \sigma_n (B_{H_n}(t_{n+1}) - B_{H_n}(t_n)).$$

- Price $p(t)$:
$$\frac{dp(t)}{p(t)} = d(t)dt + dB_{H,\sigma}(t)$$
where $B_{H,\sigma}$ is a multi-fractional process (H_t and σ_t are time-dependent) (Lévy-Véhel 1995).
- If $H_t \equiv H$ and $\sigma_t \equiv 1$, then $B_{H,\sigma} = B_H$ fractional Brownian motion.

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→ Some issues:

- Rapid Monte-Carlo simulation of price "paths".
- Estimation of the parameters.
  ↪ use of wavelets.

Wavelets

- Main context: Signals may have frequency content that varies with time. Ex: speech.
- A Fourier decomposition gives the "global" frequency decomposition.
- A wavelet decomposition gives a local characterization of the frequency contents.

$\rightarrow$ The wavelets are useful to detect changes in the multiscale character of the prices.

Cf: Haar, Stromberg, Meyer 1984, Mallat ...

The case of dyadic Haar wavelets

$\rightarrow$ Decompose the signal into components with different lengths (scales):

Question: At any given period, how much energy is there in the different scales?

The case of dyadic Haar wavelets, continued

Denote the approximation coefficients at level zero (the data) by:

$$\boldsymbol{X} = (a_0(1), a_0(2), ..., a_0(2^M)),$$

with M "dyadic" dimension. Then, at the scale j (corresponding to frequency 2^{-j}), define the approximation and difference/detail coefficients by:

$$\begin{aligned} a_j(n) &= \frac{1}{\sqrt{2}}(a_{j-1}(2n) + a_{j-1}(2n-1)) \\ d_j(n) &= \frac{1}{\sqrt{2}}(a_{j-1}(2n) - a_{j-1}(2n-1)) \ , \quad \text{for} \quad n = 1, 2, ..., 2^{M-j} \end{aligned}$$

for $j = 1, ..., M$.

$$\boldsymbol{X} \mapsto \left\{ \left\{ d_j(n), n = 1, \cdots, 2^{M-j} \right\} j = 1, \cdots, M \right\} \ \& \ a_M \, .$$

Coefficients from the continuum

If $a_0(n) = \int_{n-1}^{n} Y(t)dt$, then the detail coefficients at level j can alternatively be expressed as:

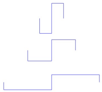
$$d_j(n) = 2^{-j/2} \int_{-\infty}^{\infty} \psi(t2^{-j} - n) Y(t) dt$$

for Y, the (quasi-continuous) data, and with the mother wavelet defined by

$$\psi(x) = \begin{cases} -1 & \text{if} \quad -1 \leq x < -1/2 \\ 1 & \text{if} \quad -1/2 \leq x < 0 \\ 0 & \text{otherwise} \end{cases} .$$

$\rightarrow$ The difference coefficients correspond to probing the process at different scales j and locations n.
$\rightarrow$ Other functions ψ give rise to different wavelet families.

Energy associated with scale



- For price $p(t) = \exp(\sigma B_H(t))$, the coefficients (at origin) of $\log P$ are:

$$d_j(0) = 2^{-j/2} \left(\int_0^{2^{j-1}} B_H(t)dt - \int_{-2^{j-1}}^0 B_H(t)dt \right)$$

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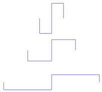

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- For energy

$$S_j = \mathbb{E}[d_j(n)^2] = \sigma^2 g(H) 2^{j(2H+1)},$$

and this gives

$$\log(S_j) = \log_2(\sigma^2 g(H)) + j(2H+1).$$

$\rightarrow$ For smooth signals, H large, there is relatively more energy in the long scales.

$\rightarrow$ H, σ determines how volatility scales in the "cascade of scales".

Scale Spectrum

For a general wavelet family we define the scale spectrum by:

$$\hat{S}_j = \frac{1}{2^{M-j}} \sum_{n=1}^{2^{M-j}} (d_j(n))^2 , \quad j = 1, 2, ..., M.$$

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• Parameter Estimation: If the underlying process has the correlation structure of fractional Brownian motion, the log-scale spectrum is affine in scale, indeed as remarked:

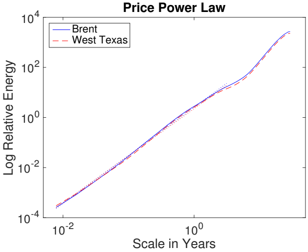
$$\mathbb{E}[\log_2(\hat{S}_j)] = \log_2(\sigma^2 g(H)) + (2H+1)j.$$

Some issues:

- What are precision-of-parameter estimates?
- How should the regression be carried out, what about inertial range?
- What about other wavelets, non-decimated case, continuous wavelet transform, optimality?
- Robustness.

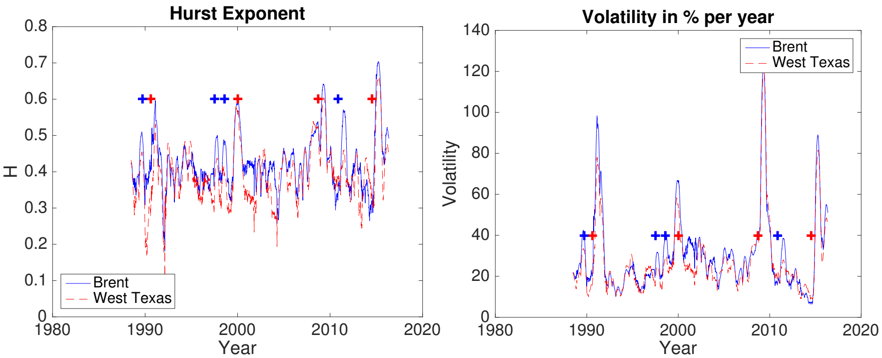
Global Scale Spectrum

- Scale spectrum for the log prices:



The "global power law" for West Texas data (red dashed line) and Brent data (blue solid line).

• A global power law (with $H = .5$) is consistent with a situation in which the Hurst exponent and volatility vary over subsegments.

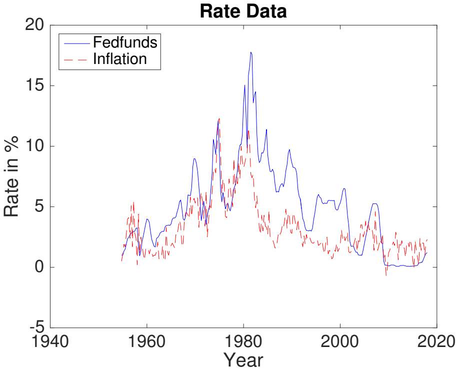


August 1990: Iraq's invasion of Kuwait, initiating a period with high volatility and a high Hurst exponent.
January 2000: fear of the Y2K bug (?), which never occurred; ending a period with relatively high volatility and high Hurst exponent.
September 2008: bankruptcy of Lehman Brothers, initiating a period with very high volatility and a high Hurst exponent.
July 2014: liquidation of oil-linked derivatives by fund managers, initiating a period with a very high Hurst exponent and high volatility.

September 1989 Fall of the Berlin wall.
June 1997 Asian financial crisis.
July 1998 Russian financial crisis.
September 2010 European debt crisis.

Scale Spectral analysis of inflation and government rate

• Quarterly data of inflation, π_n, (red dashed) and federal interest rate r_n (solid blue):



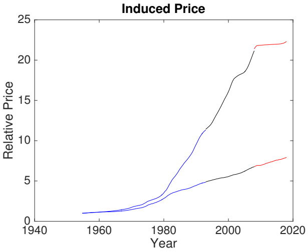
"Induced growth rate processes"

- Let the *growth rate* be defined by

$$log(p_n) - log(p_0) = \frac{1}{4} \sum_{j=1}^{n} \pi_j ,$$

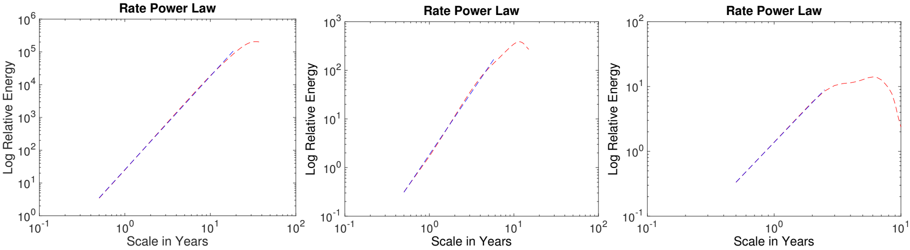
and similarly for the interest.

- 3 main periods: 1954–1992; 1992–2006; 2006–2017



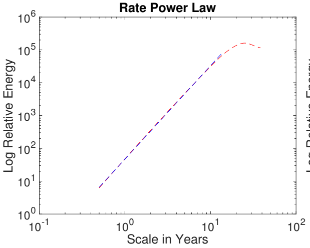
Scale Energy Distribution Interest Inflation (Cumsum)

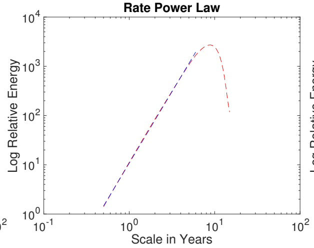
Mean: 4.1%, 2.2%, 1.6%. Hurst: 0.9, 0.8, 0.5. σ: 2.7%, 0.8%, 1.0%.
Outer scale: 19 years, 6 years, 2.5 years.

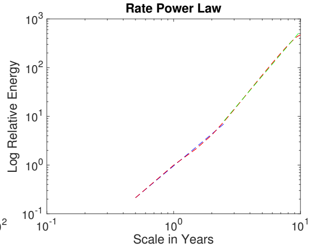


Scale Energy Distribution Fed Rate (Cumsum)

Mean: 6.3%, 4.1%, 0.5%. Hurst: 0.9, 0.9, 0.6. σ: 3.7%, 1.7%, 0.8%.
Outer scale: 19 years, 6 years, 2.5 years.







On Multiscale Correlations

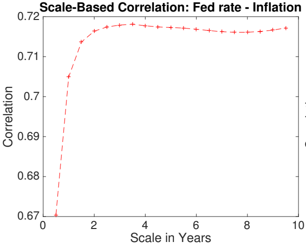
• We compute the scale-based correlation of X, Y by

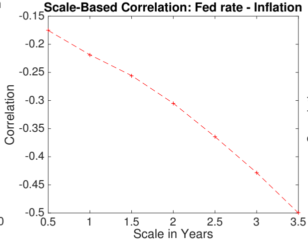
$$C^j_{x,y}(\Delta n) = Cov\left(d^x_j(n), d^y_j(n+\Delta n)\right),$$

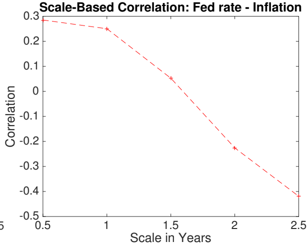
with d^x the difference (Haar) coefficients associated with the process X and with d^y the difference (Haar) coefficients associated with the process Y.

Multiscale Correlation

Scale-based correlation: periods: 1954–1992; 1992–2006; 2006–2017







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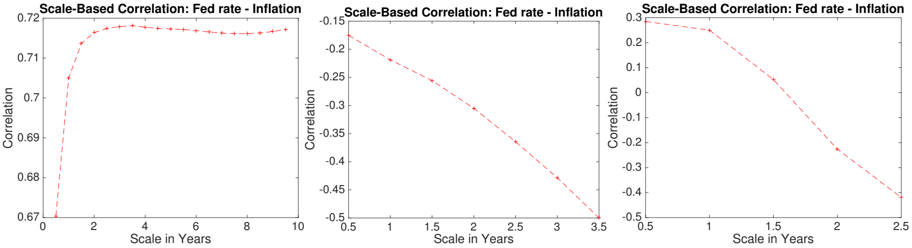


Remark on representation: $p^{(i)}(t) = p_0^{(i)} \exp(d(t) + \sigma B_H^{(i)}(t))$ with

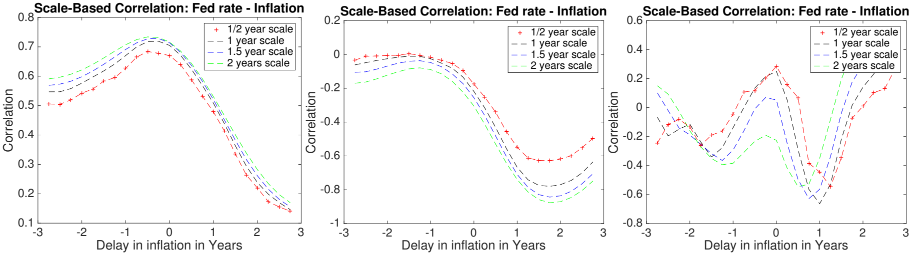
$$B_H^{(i)}(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} dB_s^{(i)}, \quad i = 1, 2$$

$$B^{(2)}(t) = \rho B^{(1)}(t) + \sqrt{1-\rho^2} W(t).$$

Varying correlation: use "spectral representation" for fBm or wavelet approach.

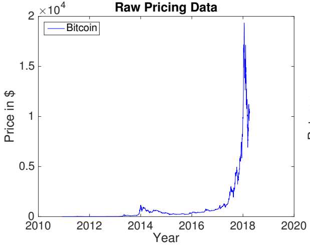
Multiscale Correlation with Offset

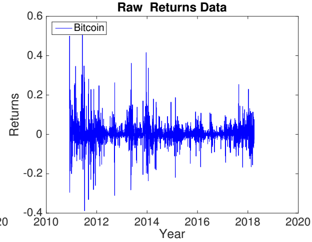
Figures for periods as above: 1954–1992; 1992–2006; 2006–2017.



$\hookrightarrow$ In period three we have a “policy response time” on about one year and a “stabilizing effect” on the yearly scale.

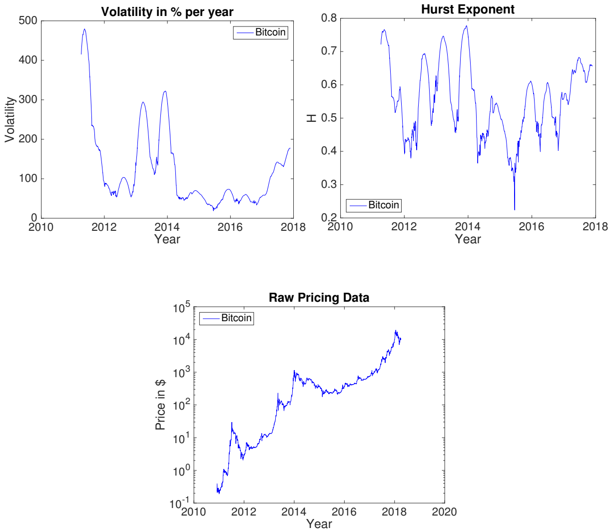
On the Bitcoin Price



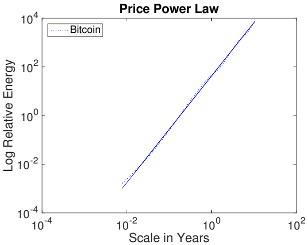


Daily Price Data From CoinDesk price page:
http://www.coindesk.com/price.

The Parameter Processes



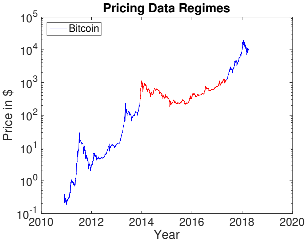
The Global Spectrum!

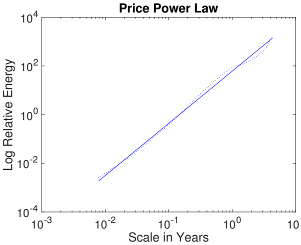


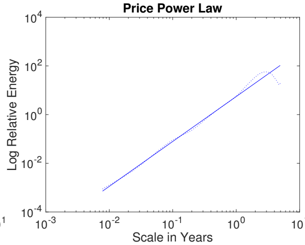
$\hookrightarrow \sigma = 143\%, H = 0.6.$

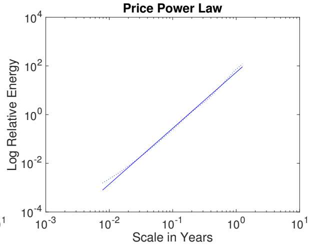
Regimes in Price Dynamics

$H = .6, .4, .7; \sigma = 171\%, 50\%, 166\%$









Final remarks

- Multi-fractal behavior can be observed in various markets.
- Important macro-economic variables seem to possess long memory
  $\rightarrow$ H matters!
- We have developed a theory for the performance of the estimator.
  $\rightarrow$ The Haar wavelets are partly superior.
- Power-law modeling is *potentially important* for regime-shift detection, prediction, pricing, hedging,...
- Viewing the market and prices through the lens of both roughness and magnitude scaling (H, σ) gives "complementary" (economic) insight about the market.
- Further issues: High- and low-frequency data, multivariate time series, periodicities, $\cdots$