# Ito calculus, Malliavin calculus and Mathematical Finance

Shigeo Kusuoka

# Mathematical Finance

Tools are given by Stochatic Analysis

**Option pricing Theory**

**Ito calculus**（**Martingale Theory** ）

**Computational Finance**

＋ **Malliavin calculus**

Historical Review on Stochastic Analysis

**Itô（1942）** Differential Equations determining a Markoff process（in Japanese）

**introduced SDE （Stochastic Differential Equation）**

**Itô（1951）** On a formula concerning stochastic differentials

**introduced Ito's formula**

**Kolmogorov（1931）** On analytical methods in probability theory

**introduced "Diffusion Equations"**

**Bachelier, Einstein:** Heat equation indirect description of Brownian motion

$\Rightarrow$ Wiener's work（1923）**Wiener measure**

**Ito's SDE** $\sigma_k : \mathbf{R}^N \to \mathbf{R}^N$, $k = 0, 1, \dots, d$

$$dX(t,x) = \sum_{k=1}^{d} \sigma_k(X(t,x))dw^k(t) + \sigma_0(X(t,x))dt$$

$$X(0,x) = x \in \mathbf{R}^N$$

**Kolmogorov's equation** $\frac{\partial}{\partial t}u(t,x) = Lu(t,x)$

$$L = \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{N} b^i(x)\frac{\partial}{\partial x^i}$$

$$a^{ij}(x) = \sum_{k=1}^{d} \sigma_k^i(x)\sigma_k^j(x), \quad b^i(x) = \sigma_0^i(x), \qquad i,j = 1,\dots,N$$

$L$ does not determine $\sigma_k$'s uniquely

**Wiener（1928）** **Homogeneopus chaos**

**Itô（1951）** **Multiple Wiener integral**

**refined Wiener's idea** **relation with Stochastic integrals**

## Ito's representaion theorem

# Stochastic integral and Martingales

**Kunita-Watanabe（1967）** **On square integrable martingales**

**Meyer, Dellacherie, . . .** **Strasbourg School**

# Analysis on Wiener Measure

## 1. Transformation of Wiener measure

**Cameron-Martin（1949）** The transformation of Wiener integrals by non-linear transformations

**Abstract version**

**Gross（1960）** Integration and non-linear transformation in Hilbert space

**Ramer（1974）** On nonlinear transformations of Gaussian measures **SDE version**

**Maruyama（1954）** On the transition probability functions of the Markov processes

**Change of measure**

**Girsanov（1960）** On transforming a certain class of stochastic processes by absolutely continuous substitution of measures

## 2. Differential Calculus in infinite dim. space with quasi-invariant measure

**Gross（1967） Potential Theory on Hilbert spaces**

**Kree, Daletsky**

**Constructive field theory（ Nelson, Glimm, Albeverio, . . . ）**

These results are **not** applicable to **SDE**

Problem on **the continuity of solutions to SDE**

# Problem on continuity of solutions to SDE

$W_0^d = \{w \in C([0,\infty); \mathbf{R}^d;\ w(0) = 0\}$

$\mu$ Wiener measure on $W_0^d$

SDE on $(W_0^d, \mathcal{B}(W_0^d), \mu)$

$$dX(t,x) = \sum_{k=1}^{d} \sigma_k(X(t,x))dw^k(t) + \sigma_0(X(t,x))dt$$

$$X(0,x) = x \in \mathbf{R}^N$$

$\sigma_k : \mathbf{R}^N \to \mathbf{R}^N$, $k = 0, 1, \ldots, d$, smooth and Lipschitz continuous

solution $X(t,x) : W_0^d \to \mathbf{R}^N$ Wiener functional

In 1970's it turned out that $X(t, x)$ is not continuous in general

In particular Lévy's stochastic area is not continuous

**Lyons** (1994) Differential equations driven by rough signal by introducing a revolutionary scheme

Differential Calculus on Wiener space

**Malliavin** (1978) Stochastic calculus of variation and hypoellip-tic operators

Analysis with respect to Ornstein-Uhlenbeck operator
(Shigekawa )

## Malliavin's integration by parts formula

$$E[F(t,x)\frac{\partial f}{\partial x^i}(X(t,x))] = E[F_i(t,x)f(X(t,x))], \qquad i = 1,\dots,N$$

if Malliavin's covariance matrix is not degenerate

Malliavin's covariance matrix is described by Lie algebra of vector fields

Probabilistic proof for Hörmander's Theorem

It was used to show the qualitative property on SDE

Practical Problem in Finance

compute $E[f(X(t,x))]$ : price of derivatives

$\frac{\partial}{\partial x^i}E[f(X(t,x)]$, $\frac{\partial^2}{\partial x^i \partial x^j}E[f(X(t,x)]$, etc. : Greeks

In 1990s, people used numerical computation method for PDE

$N$ is high ( $N \geqq 4$ sometimes )

Domains are not bounded

Monte Carlo methods or quasi Monte Carlo methods

Euler-Maruyama method: 1-st approximation

Use of Malliavin calculus

Computation of Greeks

Higher order approximation: KLNV method

Computation of $E[1_{(0,\infty)}(X^1(t,x))]$

$$\frac{1}{M}\sum_{m=1}^{M} 1_{(0,\infty)}(\tilde{X}_m) \tag{1}$$

$\tilde{X}_m$, $m = 1, 2, \ldots$, independent RV : law $X(t,x)$

$$E[1_{(0,\infty)}(X^1(t,x))] = E[\frac{\partial f_1}{\partial x^1}(X(t,x))] = E[F_1(t,x) f_1(X(t,x))]$$

$$f_1(x) = \max\{x^1, 0\},\ \ x^1 \in \mathbf{R}^N$$

$$\frac{1}{M}\sum_{m=1}^{M} \tilde{F}_m f_1(\tilde{X}_m) \tag{2}$$

$(\tilde{F}_m, \tilde{X}_m)$, $m = 1, 2, \ldots$, independent RV : law $(F_1(t,x), X(t,x))$