Modeling rough covariance processes

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From Microscopic Models to Rough Macroscopic Models

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Rough volatility

- Revolutionary modeling paradigm initiated by Gatheral, Jaisson and Rosenbaum ('18) in the already seminal paper

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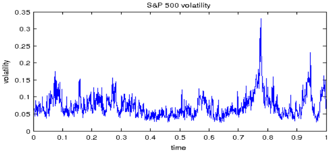
- Rough volatility: $(V_t)_{t \geq 0}$ is no longer a semimartingale, but a process whose trajectories can be rougher than the ones of Brownian motion (e.g. fractional Brownian motion with a low Hurst parameter ≈ 0.1).
- Empirical evidence comes from both
  - time series data,
  - option data.



Modeling with stochastic Volterra equations

Microstructural foundations of rough volatility (El Euch et al.) suggest stochastic Volterra equations with fractional kernels on $\mathbb{R}_+$ as models for the spot variance.

Stochastic Volterra equation

$$V_t = g(t) + \int_{[0,t)} K(t-s) dZ_s, \qquad \text{(SVE)}$$

where Z is an Itô semimartingale with differential characteristics

$$\text{drift } b(V), \text{ diffusions matrix } a(V), \text{ jump compensator } \nu(V, d\xi).$$

The initial conditon g is deterministic and the kernel $K \in L^2_{\text{loc}}$ (including singular fractional kernels $K(t) \approx t^{\alpha}$, $\alpha \in (-\frac{1}{2}, 0)$) .

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Challenges

- V is not (one-dimensional) Markovian.
- V is not a semimartingale in general, in particular if K is a fractional kernel.

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$$\lambda_t = g(t) + \int_0^t K(t-s)dN_s.$$

- Scaling λ and choosing g and K appropriately, the intensity λ converges in the long run to a rough Cox Ingersoll Ross process

$$V_t = g(t) + \int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha)}\kappa(\theta - V_s)ds + \int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha)}\sqrt{V_s}dW_s,$$

where $\alpha \in (-\frac{1}{2}, 0)$ and W a Brownian motion. Together with the log price P, this yields the rough Heston model.

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- For further convergence results of so-called "affine forward intensity models" to rough models see Gatheral & Keller-Ressel ('18).

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  - B is a Brownian motion and μ^X the random measure of the jumps;
  - $\sigma, \beta \in \mathbb{R}$ and μ a Lévy measure exhibiting a second moment;
  - K is a kernel in $L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$
  - $t \mapsto g(t)$ a deterministic function satisfying certain conditions such that V is nonnegative for all times.
- For the intensity of the Hawkes process the parameters are $\sigma = 0$, $\beta = 1$ and $\mu = \delta_1$.

Laplace Transform - Tractability

For standard finite dimensional linear Markov processes (analgously affine) it holds:

Theorem

The Fourier-Laplace transform of the affine Markov process V is

$$\mathbb{E}\left[\exp(uV_t)\right] = \exp(V_0\psi(t)),$$

where ψ is a solution of a generalized Riccati differential equation

$$\partial_t\psi(t) = \mathcal{R}(\psi(t)), \quad \psi_0 = u.$$

where $\mathcal{R}(u) = \beta u + \frac{1}{2}\sigma^2 u^2 + \int(e^{u\xi} - 1 - u\xi)\mu(d\xi)$.

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where ψ is a solution of a Riccati Volterra equation

$$\psi(t) = K(t)u + \int_0^t K(t-s)\mathcal{R}(\psi(s))ds, \quad t > 0.$$

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Remark: El Euch and Rosenbaum ('17) were the first to discover this form of the Fourier-Laplace transform in the rough Heston case, Abi Jaber et al. ('17) then for continuous affine Volterra processes.

Infinite dimensional affine processes in a nutshell

- Affine processes qualify as universal model class.
- Certain nonnegative measure-valued jumps diffusion (superprocesses) are well-known examples:
  - Dawson Watanabe process and branching Brownian motion.
  - The Riccati equations correspond here to certain non-linear PDEs, (e.g. the logarithm of the KPP equation).
  - The set of nonnegative measures on compacts is locally compact. $\Rightarrow$ standard Feller and martingale problem theory can be applied.

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- Our work:
  - We consider signed measure-valued lifts of affine Volterra processes. Here the state space is no longer locally compact $\Rightarrow$ Generalized Feller theory
  - The Volterra Riccati equations arise from the infinte dimensional Riccati ODEs associated to the signed measure valued lifts.

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- Recent related literature: Cox et al. ('21), Schmidt et al. ('20), Benth and Simonsen ('18)

Questions in the context of rough volatility

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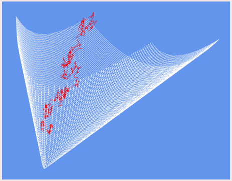
- What about the multivariate situation when we have d assets? Is there empirical evidence for rough covariance processes?
- How can we model rough (affine Volterra type) processes in the cone of positive semidefinite matrices $\mathbb{S}_+^d$, in particular a rough Wishart process? How can we exploit infinite dimensional lifts?



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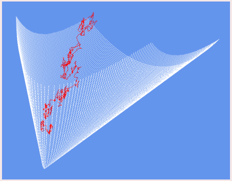

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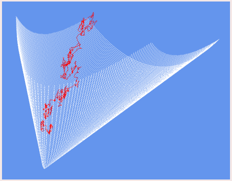


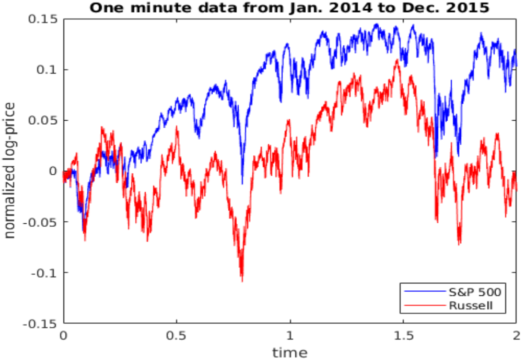
- Straightforward generalization is not possible due to the geometry of $\mathbb{S}_+^d$.
- What is an analog of the rough Heston model?

Goals and structure of this talk

1. Empirical evidence for rough covariance
2. Review of classical (non rough) multivariate affine stochastic covariance models
3. Introduction of (rough) affine Volterra-type proceses on $\mathbb{S}_+^d$
   - Pure jump processes
   - Volterra Wishart processes
   - Multivariate (rough) Volterra-Heston type models

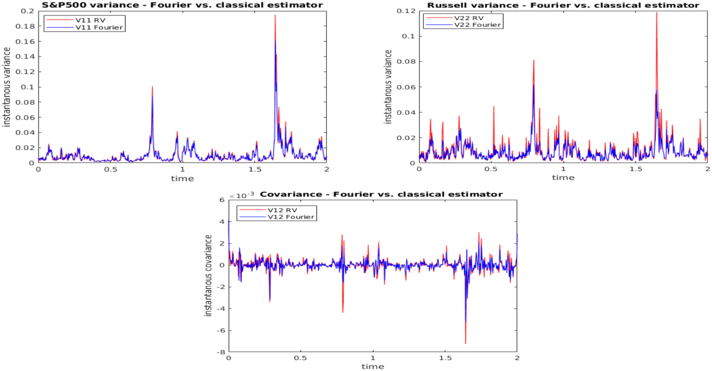
Pathwise covariance estimation - real data

- 1 minute log-price data of S&P 500 and Russell 2000 over two years



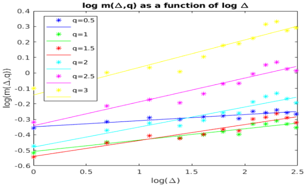
Reconstruction of the (co)variance path - real data

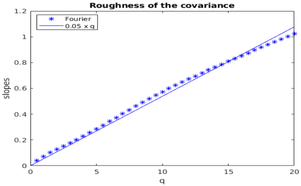
- Different realized variance estimators suggest qualitatively similar results.
- Local realized variance estimator, truncated for jumps, with a window Δ of one day versus a jump robust Fourier estimator



Estimating roughness of the spot (co)variance

- Use similarly as in Gatheral et al. ('18) a q-variation estimator of the estimated covariance: $\widehat{m}(\Delta, q) = \frac{1}{N}\sum_{k=1}^{N} |\widehat{V}^{ij}_{k\Delta} - \widehat{V}^{ij}_{(k-1)\Delta}|^q$.
- If $\frac{1}{\Delta^{qH_q}}\widehat{m}(\Delta, q) \to b_q$ in probability, then the trajectory $t \mapsto V^{ij}_t$ lies in a Besov space $\mathcal{B}^{H_q}_{q,\infty} \Rightarrow$ Hölder continuity with $H_q - \frac{1}{q}$.
- For different assets and estimators, the Hölder exponent H of the covariance is considerably smaller than of the individual variances, around 0.05 in contrast to 0.1.





- Possible conclusions: the correlation process is rougher and/or effect of asynchronous data $\Rightarrow$ Rough covariance models cannot be rejected

Insights on estimating roughness of the spot covariance

- For the Fourier spot (co)variance estimator the convergence rate is of order $n^{\frac{\gamma-1}{2\gamma}}$ where n corresponds to the number of asset price observations and $\frac{\gamma-1}{2\gamma} \in (0, \frac{H}{2H+1})$. The rougher $t \mapsto \sqrt{V_t}$, the lower the optimal convergence rate.
- The estimator noise makes it hard to identify the true roughness fully non-parametrically on the basis of 1 minute data.
- In the case with low Hurst parameter, it is difficult to separate signal from noise in a non-parametric way.
- For recent studies on different time scales see Garcin and Graselli ('20).

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- For recent studies on different time scales see Garcin and Graselli ('20).
- Assuming a more specific class of models, such as exponentials of fractional OU processes as in Gatheral et al. ('18), allows to
  - exploit further properties (e.g. autocorrelation)
  - to reject models with higher Hurst parameters on the data basis.

Classical multivariate affine stochastic covariance models

- Log price of d assets: $dP_t = -\frac{1}{2}\text{diag}(V_t)dt + \sqrt{V}_t dW_t$
- Covariance process with values in $\mathbb{S}_+^d$ (Wishart process with jumps)

$$V_t = \underbrace{V_0 + bt}_{g(t)} + \int_0^t \beta(V_s)ds + \int_0^t \sqrt{V_s}dB_sQ + Q^\top dB_s^\top \sqrt{V_s} + N_t$$

  - B is a $d \times d$ matrix of Brownian motions and $W_t = \sqrt{1-\rho^\top \rho}\widetilde{B}_t + B_t\rho$ an $\mathbb{R}^d$ valued Brownian motion correlated with B via $\rho \in \mathbb{R}^d$
  - N is a jump process with jump sizes in $\mathbb{S}_+^d$ of finite variation with compensator $M(v, d\xi) = \frac{\text{Tr}(v\mu(d\xi))}{\|\xi\| \wedge 1}$ with μ an $\mathbb{S}_+^d$-valued finite measure
  - β a linear operator satisfying admissibility conditions, $Q \in \mathbb{R}^{d\times d}$,
  - $b \succeq (d-1)Q^\top Q$ (or $b = nQ^\top Q$ if $rk(V_0) \leq n+1$, $d-1 > n \in \mathbb{N}$)

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- Tractable multivariate models that exhibit stochastic correlation, but that are not necessarily in line with empirical evidence shown above.

Affine Volterra jump processes on $\mathbb{S}_+^d$

- Let
  - $g : \mathbb{R}_+ \to \mathbb{S}_+^d$ be a some deterministic function,
  - K a (potentially fractional) kernel in $L^2(\mathbb{R}_+, \mathbb{S}_+^d)$ that can give rise to different roughness regimes.
  - N a pure jump process of finite variation with jump sizes in $\mathbb{S}_+^d$, whose compensator is $M(v, d\xi) = \frac{\mathrm{Tr}(v\mu(d\xi))}{\|\xi\| \wedge 1}$ with μ an $\mathbb{S}_+^d$-valued finite measure on $\mathbb{S}_+^d$ satisfying $\int_{\|\xi\| \geq 1} \|\xi\|^2 \|\mu(d\xi)\| < \infty$.
- Our first goal is to analyze $\mathbb{S}_+^d$-valued "intensities" of Hawkes type processes of the form

$$V_t = g(t) + \int_0^t (K(t-s)V_s + V_s K(t-s))ds + \int_0^t K(t-s)dN_s + \int dN_s K(t-s).$$

- The components of N can then be interpreted as up and downward jumps of asset prices in spirit of Rosenbaum and Tomas ('19).

Projections of processes with values in $\mathbb{S}_+^d$-valued measures

We analyze these processes by means of infinite dimensional lifts and generalized Feller processes.

Theorem (C. and Teichmann ('19))

- *Let $K, g \in \mathbb{S}_+^d$ be such that $K(t) = \int_0^\infty e^{-xt}\nu(dx)$ and $g(t) = \int_0^\infty e^{-xt}\lambda_0(dx)$ with ν, λ_0 being $\mathbb{S}_d^+$ valued measures on $\mathbb{R}_+$ satisfying certain technical conditions (ν can give rise to fractional kernels).*

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Then the above Volterra jump process is the projection, namely the total mass $V_t = \int_0^\infty \lambda(dx)$, of an affine generalized Feller process λ which takes values in the space of $\mathbb{S}_+^d$-valued measures on $\mathbb{R}_+$ of the form

$$d\lambda_t(dx) = \left(-x\lambda_t(dx) + \nu(dx)\left(\int_0^\infty \lambda_t(dx)\right) + \left(\int_0^\infty \lambda_t(dx)\right)\nu(dx)\right)dt + \nu(dx)dN_t + dN_t\nu(dx), \quad \lambda_0 = \lambda_0.$$

Laplace transform

Theorem (C. and Teichmann ('19))

Moreover, the Laplace transform of V is given by

$$\mathbb{E}[\exp(\mathrm{Tr}(uV_t))] = \exp\left(\mathrm{Tr}(g(t)u) + \int_0^t \mathrm{Tr}(g(t-s)\mathcal{R}(\psi(s))ds\right), \quad u \in \mathbb{S}_-^d,$$

where

$$g(t) = \int_0^\infty e^{-xt}\lambda_0(dx), \quad \mathcal{R}(u) = u + \int_{\mathbb{S}_+^d} (e^{Tr(u\xi)} - 1)\frac{\mu(d\xi)}{1 \wedge \|\xi\|}$$

and ψ solves the matrix Volterra equation

$$\psi(t) = uK(t) + \int_0^t \mathcal{R}(\psi(s))K(t-s)ds.$$

Hence the solution of the stochastic Volterra equation is unique in law.

Towards Volterra type Wishart processes

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- In contrast to the one dimensional case, convergence results of the previous Hawkes type process' intensities cannot be obtained.
- For the classical Wishart process we have a drift conditon that grows linearly with the dimension d. $\Rightarrow$ Crucial obstruction to infinite dimensional processes.
- If we restrict the process to take values in rank $n < d$ submanifolds of $\mathbb{S}_+^d$, the drift condition depends only on n (and the diffusion matrix).
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- Build matrix squares of Volterra OU processes taking values in $\mathbb{R}^{n\times d}$

$$X_t = g(t) + \int_0^t dW_s K(t-s)$$

with $K(t) = \int_0^\infty e^{-xt}\nu(dx) \in \mathbb{S}^d$ and W an $n \times d$ matrix of BMs.

Affine structure via infinite dimensional lifts?

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- Lift X_t to an infinite dimensional stochastic process taking values in $\mathbb{R}^{n\times d}$ -valued measures on $\mathbb{R}_+$ denoted by $Y^*(\mathbb{R}^{n\times d})$

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which takes values in $\widehat{\mathcal{E}} := \{\gamma \widehat{\otimes} \gamma \in Y^*(\mathbb{R}^{n\times d}) \widehat{\otimes} Y^*(\mathbb{R}^{n\times d})\}$ i.e., finite $\mathbb{S}_+^d$-valued, rank n, product measures on $\mathbb{R}_+ \times \mathbb{R}_+$.

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- Take as pairing between elements in $\widehat{\mathcal{E}}$ and corresponding functions $y_1 \widehat{\otimes} y_2 \in Y(\mathbb{R}^{n\times d}) \widehat{\otimes} Y(\mathbb{R}^{n\times d})$

$$\langle y_1 \widehat{\otimes} y_2, \gamma_1 \widehat{\otimes} \gamma_2 \rangle = \operatorname{Tr}\left(\int_0^\infty y_1^\top(x_1) y_2(x_2) \gamma_1^\top(dx_1) \gamma_2(dx_2) \right).$$

Infinite dimensional Wishart processes

Theorem (C. and Teichmann ('19))

The process λ is Markovian on $\widehat{\mathcal{E}}$. The corresponding semigroup is a generalized Feller semigroup. Moreover, for $y \in Y(\mathbb{R}^{n\times d})$

$$\mathbb{E}_{\lambda_0}\left[\exp\left(-\langle y\widehat{\otimes}y, \lambda_t\rangle\right)\right] = \exp(-\varphi_t - \langle \psi_t, \lambda_0\rangle),$$

where ψ and φ satisfy the following matrix-valued Riccati PDEs namely $\psi_0 = y\widehat{\otimes}y$ and $\partial_t\psi_t = R(\psi_t)$ in the mild sense with $R : \widehat{\mathcal{E}}_* \to \widehat{\mathcal{E}}_*$ given by

$$\begin{aligned} R(y\widehat{\otimes}y)(x_1, x_2) &= -(x_1 + x_2)y(x_1)\widehat{\otimes}y(x_2) \\ &\quad - 2\int_0^\infty \int_0^\infty y(dx_1)\widehat{\otimes}y(dx)\nu\widehat{\otimes}\nu(dx, dy)y(dy)\widehat{\otimes}y(dx_2) \end{aligned}$$

and $\varphi_0 = 0$ and $\partial_t\varphi_t = F(\psi_t)$ with $F : \widehat{\mathcal{E}}_* \to \mathbb{R}$ given by

$$F(y\widehat{\otimes}y) = n\langle y\widehat{\otimes}y, \nu\widehat{\otimes}\nu\rangle.$$

Volterra Wishart process

- The Volterra Wishart process defined as
$$V_t = X_t^\top X_t = \int_0^\infty \int_0^\infty \lambda(dx_1, dx_2)$$
is thus a projection of an infinite dimensional affine process. Its Laplace transform can be computed by setting $\psi_0 = Id$. (c.f. also Abi Jaber ('20))
- Its dynamics can be expressed as follows
$$V_t = g(t)^\top g(t) + n \int_0^t K(t-s)K(t-s)ds + \int_0^t K(t-s)dW_s^\top \mathbb{E}[X_t|\mathcal{F}_s]ds + \int_0^t \mathbb{E}[X_t|\mathcal{F}_s]^\top dW_s K(t-s),$$
- The marginals of V are Wishart distributed as they are squares of Gaussians.
- It is not of standard Volterra form as it depends on $(\mathbb{E}[X_t|\mathcal{F}_s])_{\{s \le t\}}$.
- Via a Brownian field representation it can however be expressed as a path functional of $(V_s)_{\{s \le t\}}$ but not only on the state V_t.

Multivariate (rough) Volterra-Heston type models

- Log-price process of d assets:

$$dP_t = -\frac{1}{2}\mathrm{diag}(V_t)dt + X_t^\top d\widetilde{W}_t,$$

where $\widetilde{W}$ is an n-dimensional Brownian motion given by $\widetilde{W}_t = \sqrt{1-\rho^\top\rho}\widetilde{B}_t + W_t\rho$ with $\rho \in \mathbb{R}^d$ and $\widetilde{B}_t$ an n-dimensional independent Brownian motion.

Theorem (C. and Teichmann ('19))

The process (λ_t, P_t) is a Markov process on $(\widehat{\mathcal{E}}, \mathbb{R}^d)$. Moreover, it is affine in the sense that for $(y, v) \in (Y(\mathbb{R}^{n\times d}), \mathbb{R}^d)$

$$\mathbb{E}_{\lambda_0,P_0}\left[\exp\left(-\langle y\widehat{\otimes}y, \lambda_t\rangle + \langle iv, P_t\rangle_{\mathbb{R}^d}\right)\right] = \exp(-\varphi_t - \langle\psi_t, \lambda_0\rangle + \langle iv, P_0\rangle_{\mathbb{R}^d}),$$

where ψ satisfies an Riccati differential equation in the space of matrix valued functions and $\varphi_t = n\int_0^t \langle\psi_s, \nu\widehat{\otimes}\nu\rangle ds$.

Conclusions

- Review of one-dimensional rough volatility models
- Some empirical evidence for rough covariance
- $\mathbb{S}_+^d$-valued affine jump processes that can serve similarly well as covariance models in particular in view of Hawkes processes and microstructural foundations
- Construction of an infinite dimensional Wishart processes and (rough) Volterra Wishart processes
- Multivariate rough Heston model
- Future work
  - True microstructural foundations for these models
  - Multivariate quadratic Hawkes models - infinite dimensional lifts

Thank you for your attention!