How to Flee Along a Straight Line*

Tracking Self-Repelling Random Walks

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Summary

This text focuses on the path of a particular random walk, that, for example, of a fugitive who is trying to escape his pursuers, but who can only move on a real line. The aim of the fugitive is to find the continuous path that leaves behind the least information possible. This problem is addressed within the framework of the theory of random walks. Different kinds of random walks are presented, starting with the well-known Brownian motion. The scaling limit of the self-repelling random walk is then explained before examining the self-repelling process, which provides an optimal strategy.

Introduction

This text focuses on the path of a particular random walk, that, for example, of a burglar who is trying to escape the forces of law and order, but who can only move on a real line. The flight therefore takes place in a one-dimensional world. The pursuers know the number of times the burglar has visited each place (thanks to hotel bills, for example, or the location of his burglaries), but they have no information about the precise dates of these visits. The aim of the fugitive is to find the continuous path that leaves behind the least information possible.

This problem can be addressed in game theory, but the present text is based on the theory of random walks, and more precisely on what are called self-repelling random walks. I start by presenting different kinds of random walks, before focusing in section 2 on the scaling limit of the self-repelling random walk, known as "true self-repelling motion", which is analogous to Brownian motion for the simple random walk. In section 3, I briefly give some properties of self-repelling walks. I examine the optimal strategy for a clever fugitive in section 4: how can he make his flight safer?

1. Random walk

a. Definition

The random walk is one of the most studied subjects in probability theory. It is a very simple thing. How is it defined?

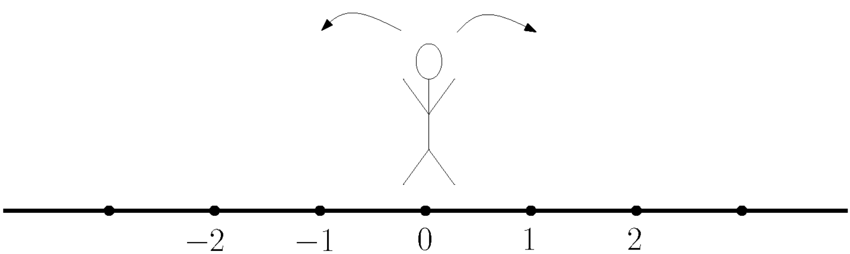


Figure 1: A random walker at his starting point

Here, the walker starts from zero and can move on the integers in dimension 1. Any other initial condition can be chosen and the discussion can easily be generalized to higher dimensions.

The random walk is constructed by recurrence. Let $(S_k)_{k \in \mathbb{N}}$ denote the successive positions of the walk. Let $S_0 = 0$. Once the walk has been defined up to time n, the walker moves according to the following law at $n + 1$: he can only jump onto the nearest neighbours, and his position at time $n + 1$ is written:

$$S_{n+1} = S_n + \varepsilon_n \text{ where } \varepsilon_n = \pm 1.$$

The move ε_n is random and may depend on the entire past, that is, all the positions visited before time n (all the $(S_k)_{k \leq n}$).

A random walk can model many different things: for example, a price changing over time, or the position of a particle in a fluid in three dimensions. In the case where the increment ε_n only depends on S_n, and not on all the positions before n, the walk is said to be Markovian. Under this hypothesis, many probabilistic tools are available to analyse the walk, but it is a very restrictive hypothesis: *a priori* ε_n really does depend on the whole past and not simply on the position at the present moment.

b. Simple random walk and Brownian motion

The most elementary example of a Markovian random walk is that of the simple random walk: a coin is tossed to choose a direction independently at each step, that is, $\varepsilon_n = 1$ with probability $1/2$ and $\varepsilon_n = -1$ with probability $1/2$ for every time n. Thus, the direction chosen does not depend on either the position at time n or the past.

The first question that naturally arises concerns the order of magnitude of S_n when n is very large. The central limit theory provides the answer: $S_n \approx \sqrt{n}$. So this case is very different from a walker who chooses always to move to his right, for example, for whom $S_n = n$, or a walker who oscillates between 0 and 1 and whose displacement is of the order of 1. Here, we are somewhere between the two.

Is it possible to refine this result by examining not only the position at time n, but also the scaling limit of the whole trajectory of this walk?

Let us start by looking at what happens in the following diagrams:

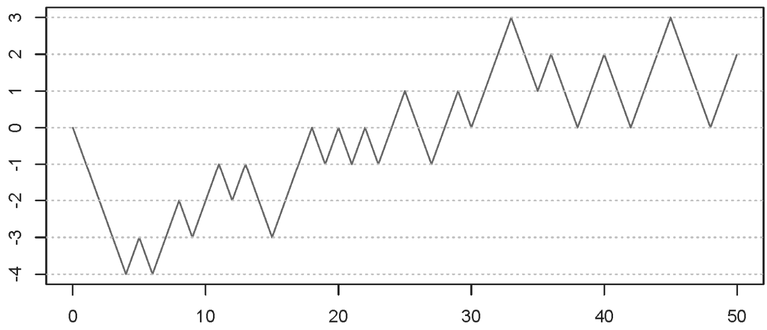


Figure 2: Simulation of a simple random walk, the trajectory $n \mapsto S_n$

The above figure represents a simulation of the simple random walk for 50 steps. The time is plotted along the *x* axis and the position of the walk along the *y* axis. The walk is considered to move along the *y* axis. For a large number of steps (10,000), the diagram looks like this:

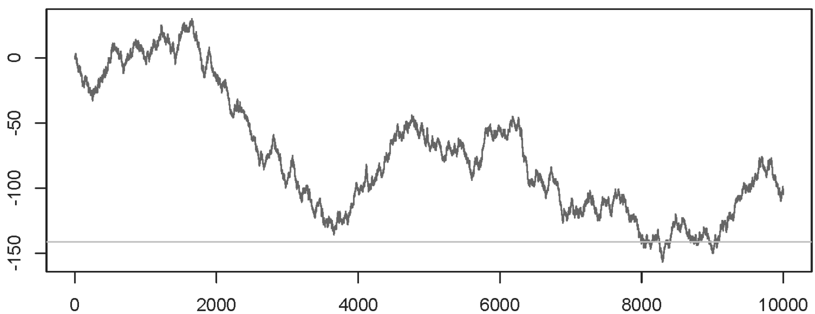


Figure 3: Simulation of a simple random walk, the trajectory $n \mapsto S_n$

Something very erratic appears. The scale is chosen in order to obtain a non-trivial object: there are 10,000 steps along the *x* axis, and the *y* axis roughly covers the interval $[-150, 50]$.

If we had chosen a scale 10,000 / 10,000, we would therefore have seen approximately a straight line (we would hardly see the fluctuations between 50 and -150). The choice of scale is therefore very important.

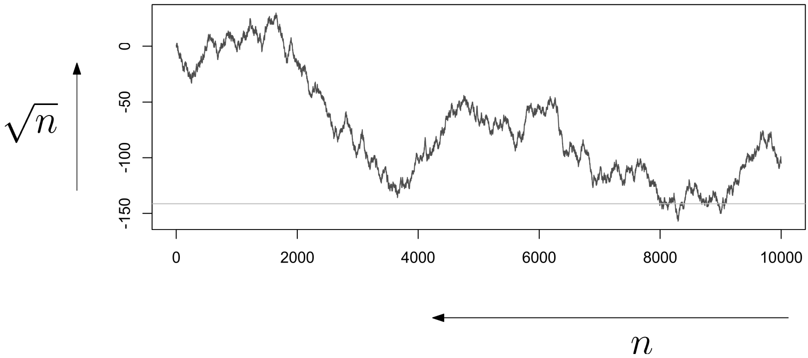


Figure 4: Rescaling a simple random walk

As in the above figure, we take an ever larger number of steps n, adjusting the scale of the walk by $\sqrt{n}$. This is the correct order of magnitude, because when $t = 1$, we find $S_{[nt]}/\sqrt{n}$, around the magnitude of 1. In the figure, this consists in dividing the *x* axis by n and the *y* axis by $\sqrt{n}$ when n increases. Is there convergence towards a limit when n tends to infinity? The answer is yes: the random function $t \mapsto S_{[nt]}/\sqrt{n}$ converges towards the Brownian motion, a continuous random fractal curve. This is the most natural random curve possible, because it corresponds to a uniform choice among all the possible continuous curves. If we spontaneously draw a continuous random function in the plane, the result should look like this:

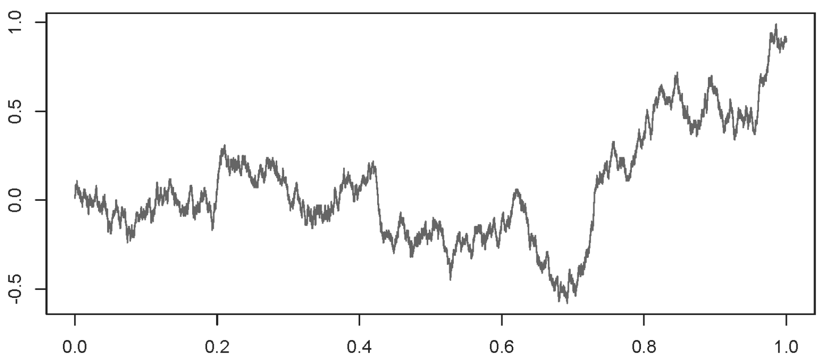


Figure 5: Brownian motion

It is a fractal curve: if we magnify a segment of it, we obtain the same curve (in law).

What is the interest of this scaling limit? The random walk is very easy to define. We could perfectly well imagine studying it when n is very large without any reference to Brownian motion, which appears, at first glance, to be more complicated to manipulate. Paradoxically, Brownian motion has much simpler properties than the random walk. In particular, it has the property of scale changing: for all $a \in \mathbb{R}$, $t \mapsto B_{at}$ follows the same law as $t \mapsto \sqrt{a}B_t$, that is, if we compress the Brownian motion by a factor of a along the *x* axis, it is equivalent (in law) to expanding the *y* axis by $\sqrt{a}$, as shown in the figure below.

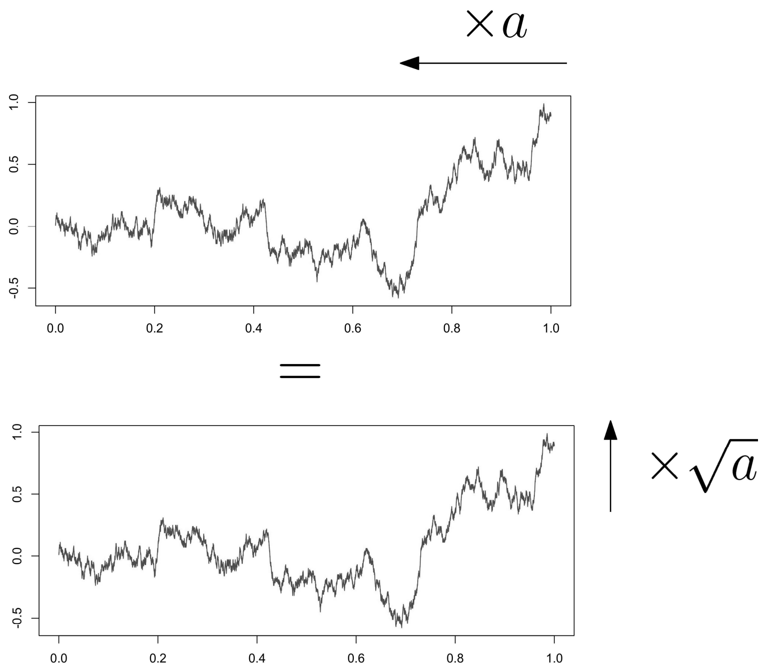


Figure 6: Self-similarity

The scaling property is very useful for calculations, and it is not verified in the case of simple random walks.

Another reason to study Brownian motion rather than the simple random walk is because it is universal. It corresponds to the scaling limit of a large number of random walks, not just the simple one. Brownian motion therefore provides information about many discrete models.

More generally, probabilists like working on scaling limits, because it allows them to use stochastic calculus to calculate various exponents. Over the last few years, this approach has enjoyed great success. In particular, the mathematicians Wendelin Werner and Stanislav Smirnov were recently awarded Fields medals for results obtained on the scaling limits of interfaces in statistical physics. Brownian motion is an essential tool for understanding these very complex interfaces.

2. Self-repelling random walk

a. Definition

What happens when the random walks are more complicated, as in the case that interests me here in this text, the "self-repelling" random walk? Here we will consider a walker who prefers travelling through the edges least visited in the past. Unlike the simple random walk, this one is non-Markovian.

To construct this walk, we must define the amount of time spent on the edges. Let (X_n) be a random walk on whole numbers. For all times $n \in \mathbb{N}$, and all edges $e = \{x, x+1\} (x \in \mathbb{Z})$, we define:

$$l(n, e) \coloneqq \#\{k \in \{0, \dots, n-1\}, \{X_k, X_{k+1}\} = e\},$$

which indicates how many times the walk has passed through the edge e (moving from left to right or from right to left) before time n.

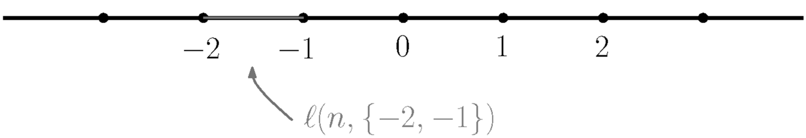


Figure 7: An edge $e = \{-2, -1\}$ and the function $l(n, e)$

The walk (X_n) always prefers turning towards new places. As far as possible, it avoids the locations it has already visited, like a tourist, for example, who is strolling along the real line and who prefers to discover previously unseen sites. This is no more than a preference, however. Sometimes, the walk returns to a point that has been visited many times before.

How is the self-repelling walk defined? If, at time n, the walk is at position X_n, we denote by l_n^+ the time spent on the edge just to the right of X_n and by l_n^- the time spent on the edge just to the left.

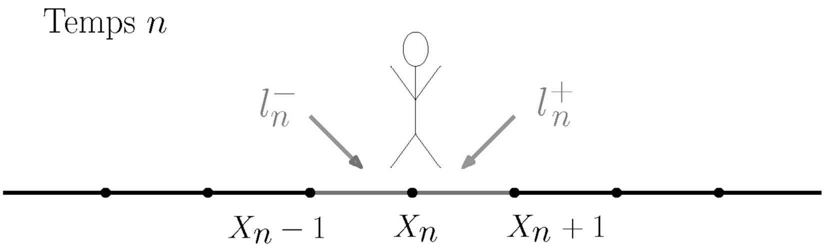


Figure 8: The self-repelling walker at time n

The probability of leaving to the right is an exponentially decreasing function of the difference between the time spent on the left and the time spent on the right:

$$\begin{aligned}\mathbb{P}(X_{n+1} = X_n + 1) &= 1 - \mathbb{P}(X_{n+1} = X_n - 1)\\ &\propto \exp\bigl(-\beta(l_n^+ - l_n^-)\bigr)\end{aligned}$$

where the parameter β is a positive fixed real number.

Thus, the more time spent on the right compared to the left, that is, the greater the difference $l_n^+ - l_n^-$, the lower the probability of returning to the right, and vice versa.

Now we can ask the same questions as we did for the simple random walk. What is the order of magnitude of X_n when n is very large? This question is not trivial: does the behaviour remain the same as in the simple random walk? Is there a correctly renormalized scaling limit $t \mapsto X_{[nt]}$? Is there then convergence towards Brownian motion or towards another process?

b. A competing model: polymers

Before answering these questions, I will sketch an outline of the history of self-repelling random walks. They were introduced by the physicists Daniel Amit, Giorgio Parisi and Luca Peliti in 1983,[1] as an original approach to modelling polymers. To define them, instead of choosing the steps one by one, by recurrence, as for a random walk, the whole trajectory is chosen on one go: $\overline{X_n} = (X_1, \dots, X_n)$.

A model of a polymer can then be written as follows:

$$\mathbb{P}(\overline{X_n} = (x_1, \dots, x_n)) \propto \exp\left(-\frac{1}{T} H(x_1, \dots, x_n)\right)$$

where $H =$ energy and $T =$ temperature.

Thus, the probability of choosing a certain trajectory, denoted by $(x_1, \dots, x_n)$, is proportional to $\exp -\frac{1}{T}$, where T is the temperature of the system, multiplied by the energy of the path whose probability we want to measure. We can choose any energy function, and the most probable configurations are the ones that require low levels of energy. If the temperature is too high, there is a lot of disorder, and energy only plays a small role. If, on the contrary, the temperature is very low, the role of energy becomes predominant.

One possible model of polymers is a model of "weakly" self-avoiding trajectories, where the energy is equal to the number of self-intersections of the path. To know the probability of a certain trajectory, we must look at the number of self-intersections. The larger this number is, the greater the amount of energy, and the less often the path is chosen. The tendency is therefore towards a path that rarely self-intersects.

The low temperature limit, when $T \to 0$, is known as the "strictly" self-avoiding walk[2] where there can no longer be any self-intersections. We make a uniform random choice between the trajectories that do not self-intersect, because a self-intersection costs too much – it entails an "infinite" increase in energy. The self-

[1] Amit, D.J., G. Parisi, and L. Peliti (1983), "Asymptotic Behavior of the 'True' Self-Avoiding Walk", *Physical Review B*, 27(3), pp. 1635–45.
[2] Even if it is not a random walk as defined above!

avoiding polymer model is of no interest in dimension 1, because the walker chooses to go to the left or the right for his first step and then always continues in the same direction in order to avoid crossing his past trajectory. In higher dimensions, the model becomes much more interesting and is much studied.

"Real" random walks, defined recursively, produce compatible families of laws over time (adding each step one by one), whereas self-repelling walks do not form such families. When the self-repelling polymer is drawn up to time n, it is difficult to know how to continue it to obtain the polymer of length $n + 1$: we have to draw the whole trajectory again!

Here is an illustration of polymers in dimension 2:

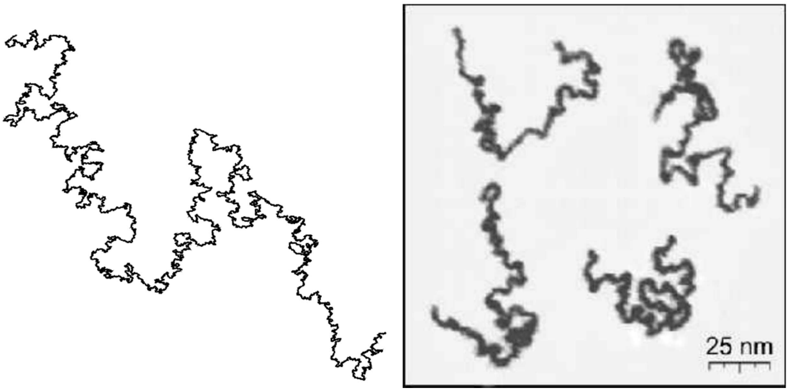


Figure 9: Polymers in dimension 2: self-repelling walk and the appearance of real linear polymer chains

The left-hand side shows the model of the self-avoiding polymer, chosen uniformly. The right-hand side shows polymer chains observed in the laboratory. We can see similarities between the two pictures.

c. Mathematical results on the self-repelling walk

Now let us return to self-repelling walks and the dimension 1. The first rigorous mathematical result was obtained by the probabilist Bálint Tóth in 1995.[3] It demonstrates that the order of magnitude of the displacement of the walker at time

[3] "'True' Self-Avoiding Walk with Bond Repulsion on $\mathbb{Z}$: Limit Theorems", *The Annals of Probability*, **23** (1995), pp. 1523–56.

n is $X_n \approx n^{2/3}$. It is therefore greater than $\sqrt{n}$ (the walk is said to be "super-diffusive"), but it is not ballistic (that is, of the order of n). This exponent $2/3$ is particularly interesting because it appears in many other models in physics.

How can we answer the second question about the scaling limit of the walk? If we normalize our walk by $n^{2/3}$, is there convergence towards a random curve, and is it still a case of Brownian motion?

There is indeed convergence, not towards Brownian motion but towards what is called the "true" self-repelling process. The figure below shows a simulation of the simple random walk (between the two horizontal straight lines $-\sqrt{n}$ and $\sqrt{n}$) and the self-repelling random walk (which moves roughly as far as $-1.5 \times n^{2/3}$).

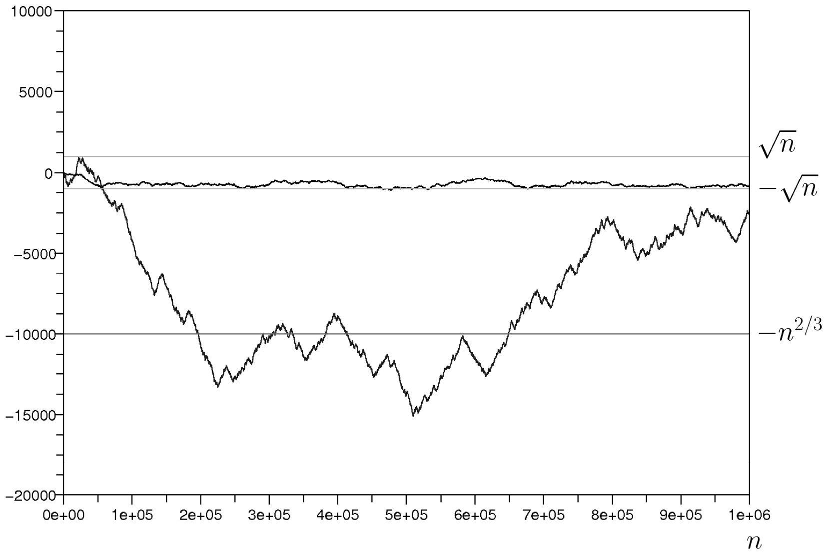


Figure 10: Comparison of the simple random walk and the self-repelling walk

We have used the appropriate scale for the self-repelling walk, with the order of magnitude $n^{2/3}$. Consequently, a non-trivial curve appears. Note that the simple random walk is located between $-\sqrt{n}$ and $\sqrt{n}$, and at this scale, it fluctuates very little. Compared with the self-repelling walk, it looks like a straight line.

This scaling limit was proved by Bálint Tóth and Wendelin Werner in 1998.[4] To understand the main idea of their proof, we will now introduce another, simpler model of self-repelling random walk.

d. The idea of Werner and Tóth's proof with a toy model

Let us present a toy model that makes it possible to understand better the self-repelling walk, although it seems of little interest at first glance. In fact, this model is crucial, because it gives the intuition for constructing the scaling limit of our self-repelling walk.

First, we introduce a particular initialization of the time spent on the edges for our walk: instead of starting from a time spent on the edges equal to 0 everywhere (which is the most natural), our function oscillates between 0 and 1. We say that the time spent on the edge $\{0,1\}$, at the initial moment, is equal to 1; the time spent on the edge $\{1,2\}$ is equal to 0, and then 1, 0, 1, 0, and so on. For the negative axis, the time spent on the edge $\{0,-1\}$ is equal to 1, and then 0, 1, 0, and so on (see the figures below).

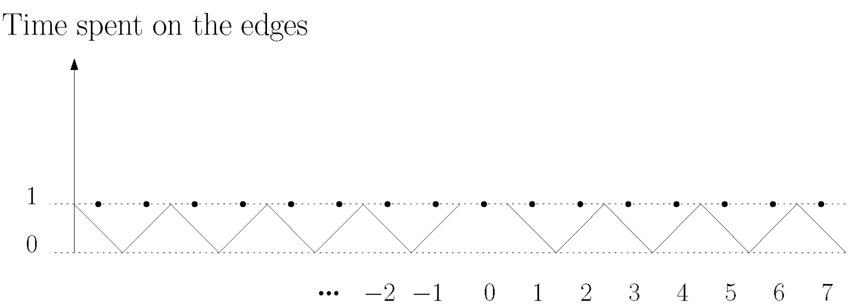


Figure 11: The initialization of the time spent on the edges is the following: $l(0,e) = 1$ or 0 depending on the edge e

[4] "The True Self-Repelling Motion", *Probability Theory and Related Fields,* 111 (1998), pp. 375–452.

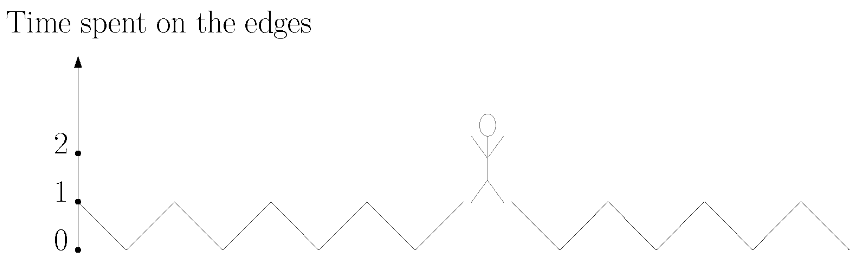


Figure 12: Realization of the walk, phase 1, $t = 0$

The walk starts at $X_0 = 0$. Starting at 0, the rules are now as follows: if the walk is constructed up until time n, then for time $n+1$, the walk will look at the number of times it has visited the neighbouring edges. These two quantities are denoted by l_n^+ and l_n^- as previously for the classic self-repelling walk. If they are equal, that is, if $l_n^+ = l_n^-$, there is no preference. We simply flip a coin to choose the next destination. If they are different, such that, $l_n^+ < l_n^-$ or $l_n^+ > l_n^-$, the walk chooses the less-visited edge deterministically (there is no randomness in this case). This produces a very simple model of self-repelling random walk: from the moment that the walker has spent less time on one side than on the other, the less-visited side will automatically be preferred.

The figure below is a representation of what happens for the first few steps:

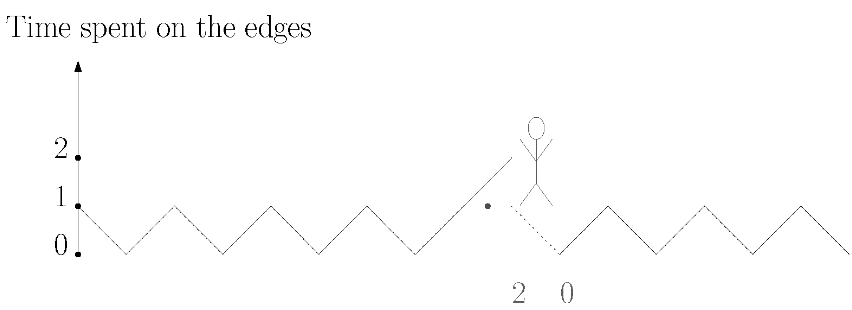


Figure 13: Realization of the walk, phase 2, $t = 1$

At the initial moment, by construction, we have spent time 1 on the left and on the right. These lengths of time are equal, so we flip a coin to choose the next direction. If the coin toss tells us to go to the right, then at time $t = 1$, the walker has spent time 2 on the edge $\{0,1\}$ and is now at the position $X_1 = 1$ (see Figure 12). We then compare the time spent on the edges $\{0,1\}$ and $\{1,2\}$, which are equal to 2 and 0 respectively.

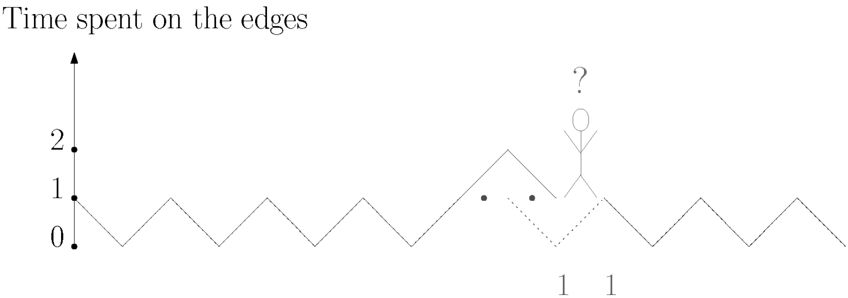


Figure 14: Realization of the walk, phase 3, $t = 2$

Deterministically, the walk chooses the less-visited place. It will therefore turn to the right and $X_2 = 2$ (Figure 14). At the moment $t = 2$, the length of time spent on each side is equal, (to the value "1"), so we toss a coin to decide which way to go at time 3, and so on.

The figures represent the progression of the walker and the time he has spent on each of the edges before the present moment. We also associate a height H_n with the walk, corresponding to the time spent at each place visited. The points indicate the places already visited and the stick figure is placed at the present position and height of the walker.

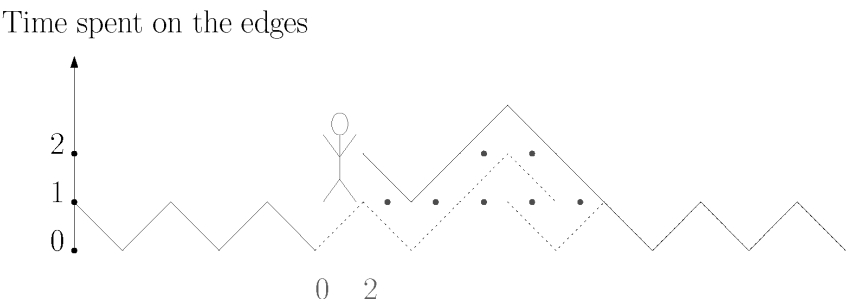


Figure 15: Realization of the walk, phase 4, $t = 7$

We thus obtain a walker who moves in a surprisingly complex way, thanks to the chosen initialization of time spent on the edges.

Why choose this particular initialization? We choose it, because in this way, we obtain a non-trivial progression and, at the same time, the flattest possible initialization (to approximate the natural one where the time spent is equal to zero everywhere).

To understand the progression from time n to time $n+1$, it is useful to represent not only the walk, but also the time spent on each edge. Are we going to flip a coin, or are we going to move to the left or right deterministically? To know this, we must look at the combined law of the position and of the time spent on each edge: $(X_n, l(n,\cdot))$.

Note that the self-avoidance in this model (as for the previous self-repelling walk) acts locally: the walk is said to be "myopic". It only looks at the time spent on the nearest edges.

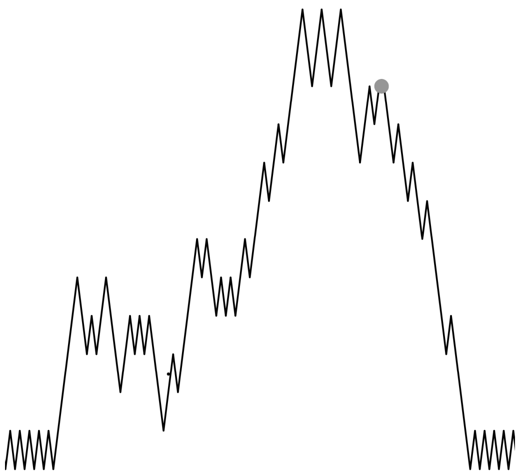


Figure 16: Time spent on the edges at moment $t = 475$ and the position of the walker at this moment (large dot)

The above graph, for example, represents the time spent on the edges for a self-repelling walker up until the moment $t = 475$. It looks as though the walk should probably move to the right for the next step (it has spent a lot less time on the right than on the left, where there is a crevice to cross).

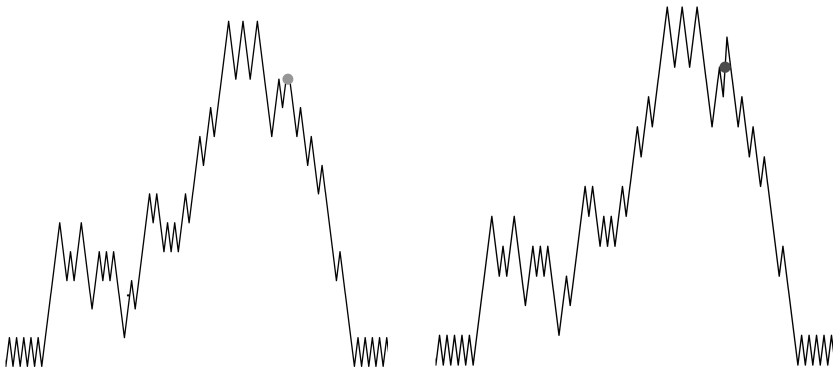


Figure 17: Time spent on the edges at the moments $t = 475$ and $t = 476$ and the positions of the walker (large dots). Note that the walker has moved to the left in this illustration.

Locally, however, it has spent as much time on the right as it has on the left. It therefore tosses a coin to decide where to move next, and there is a one-in-two chance that it will go to the left. This model therefore presents a particular form of self-avoidance.

The next step is to construct a labyrinth in dimension 2. We are thus adding a dimension to our problem to understand it better!

Let us start with a black line and a grey line, which recall the initial conditions for the time spent on the edges in our toy model:

Figure 18: Construction of the discrete network, phase 1

We then define two grids as in the figure below: there are grey points and black points, such that the union of all the points produces the grid $\left(\frac{1}{2} + \mathbb{Z}\right) \times \mathbb{N}$, and the colour is chosen according to the parity of the sum of the *x* axis and the *y* axis.[5]

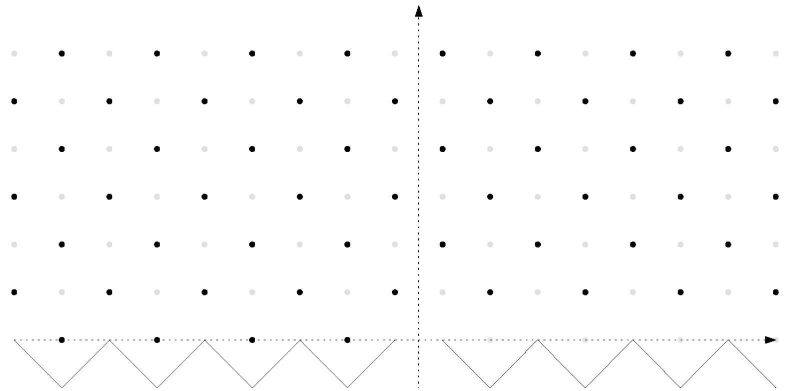


Figure 19: Construction of the discrete network, phase 2

Now, each black point is the starting point of a simple random walk (in dimension 1, the *x* axis representing the time and the *y* axis the position) reflected on the grey line until it touches the boundary (the initial black condition); at that moment, it stops:

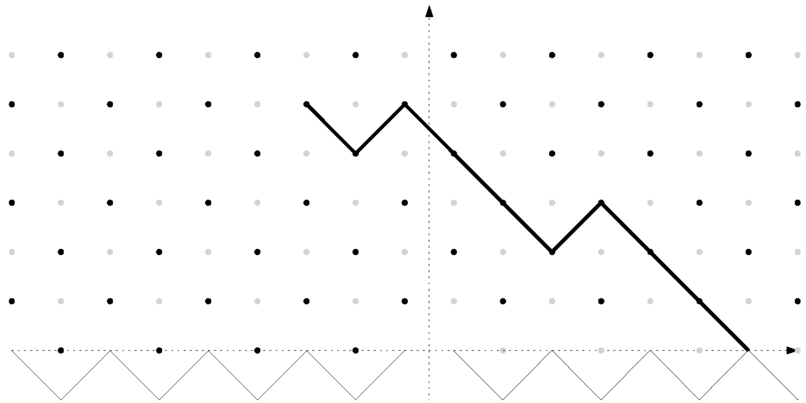


Figure 20: Construction of the discrete network, phase 3

[5] The black points (respectively grey) are situated on positions (x, y) such that $x + y - 1/2$ is even (respectively odd).

That is how the walk progresses on the black grid. Now let us take another point on the black grid, no matter which point as long as it has not been touched by this walk. From this new point we draw another random walk, independent of the first one, which comes to an end as soon as it touches something that has already been plotted.

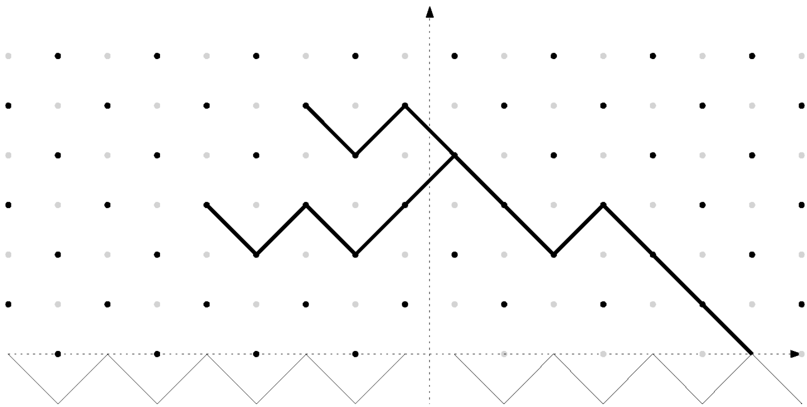
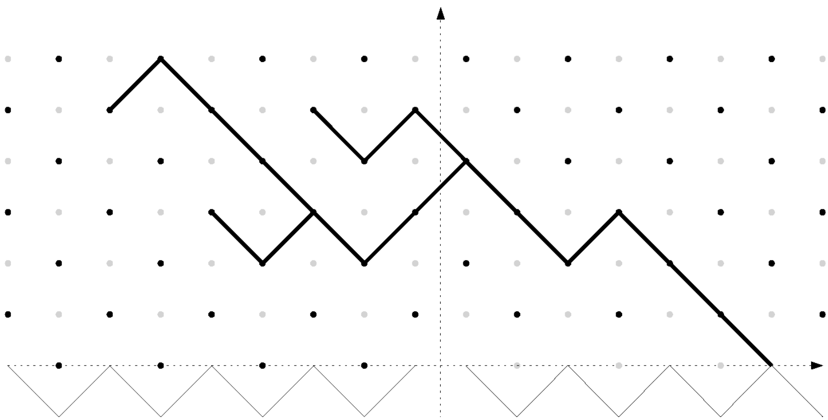


Figure 21: Construction of the discrete network, phase 4

We repeat this exercise for all the points on the black grid. We obtain a structure of coalescing random walks in the upper half plane, as in the figure below:



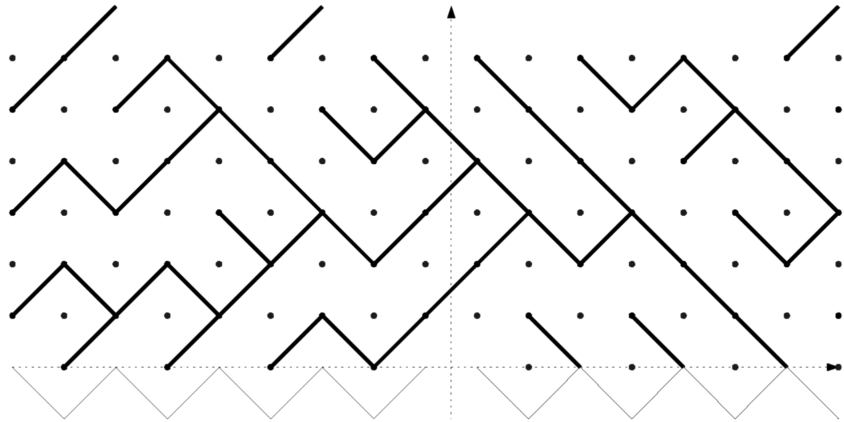


Figure 22: Construction of the discrete network, phases 5 and 6

We can then connect the grey points together, taking care not to cross the black lines. The grey structure is dual to the black structure:

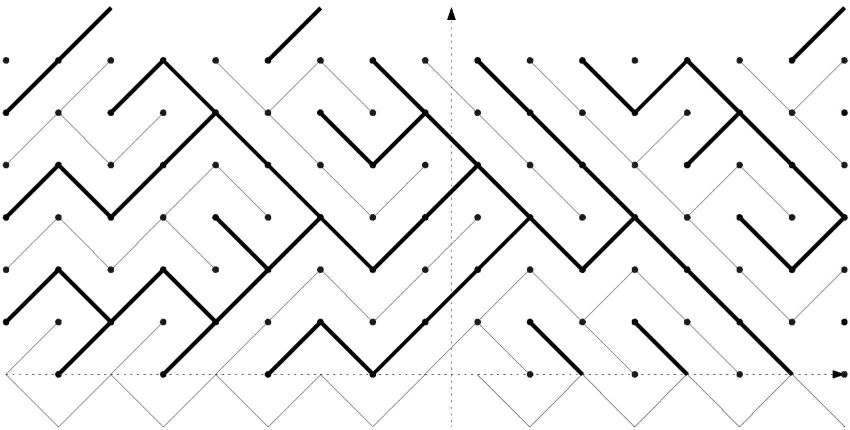


Figure 23: Construction of the discrete network, phase 7

This marks out a sort of labyrinth: we can now start from the point $(0,0)$ (the arrow in bold at the bottom of the figure below) and begin to circulate around the maze.

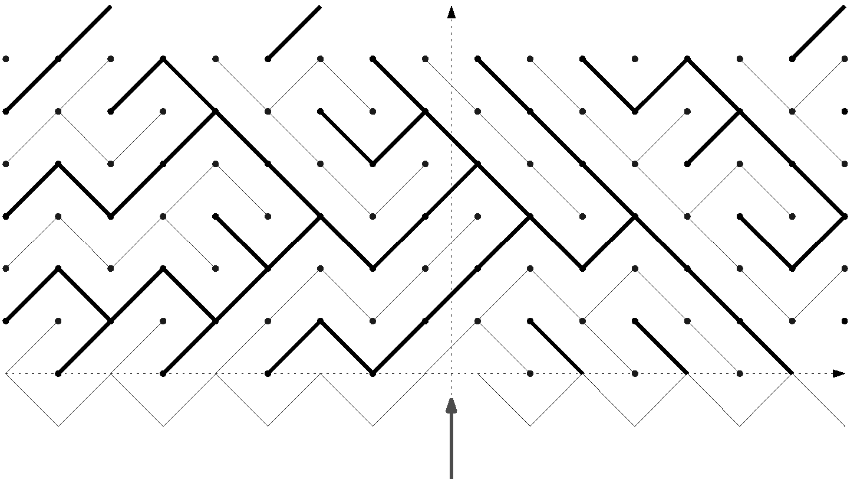


Figure 24: Walk going through the labyrinth, phase 1

The beginning of the walk through the labyrinth:

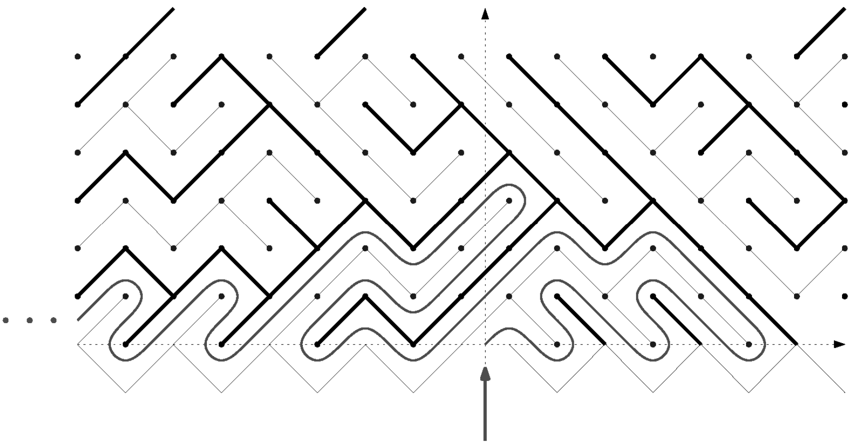


Figure 25: Walk going through the labyrinth, phase 2

What is the connection with the self-repelling walk? It so happens that the walk going through the labyrinth has the same law as the self-repelling walker (endowed with height H_n such that he does indeed progress in the upper half plane) in the toy self-repelling model (for an illustration of this fact, compare Figures 15 and 24: we can see the beginning of the labyrinth in the dotted line in Figure 15).

So the two models are actually one and the same! This observation is central for constructing the limit law of the toy model walker.

To obtain the scaling limit of the toy model self-repelling walker, we construct a continuous analog of the discrete labyrinth. This is possible, because it is defined with simple random walks, for which the scaling limit is well-known: it is the Brownian motion we saw earlier. So we replace the simple random walks of the labyrinth by Brownian motions (which is not easy because the Brownian motions will start from every point in the upper half plane). The walk is then performed through this continuous labyrinth: this is the walk that corresponds to the scaling limit of our toy model walker. We call this limit the (true) **self-repelling motion**.

Note that the same kind of idea (looking at the outline of the trees defining labyrinths) allowed Gregory Lawler, Oded Schramm and Wendelin Werner[6] to construct the scaling limit of the uniform spanning tree, a very well-known statistical physics model, and to prove its conformal invariance.

3. Some properties of the true self-repelling motion

What can we say about the scaling limit of self-repelling walks, this self-repelling process?

First of all, what is a process? A "process" $(X_t)_{t\geq 0}$ is a continuous random function that associates a position X_t with each real positive time t.

How can we see whether it is self-repelling? We no longer have edges, and we can no longer "count" how much time is spent on each point (in fact, an *infinitesimal* time is spent on each position x!). Nevertheless, for some processes, it is possible to find a way to measure the time spent on each point, and we call this the **local time** $L_t(x)$. As in the case of the random walkers discussed above, for the processes that allow a local time, we can consider the $1+1$ dimensional process:

$$(X_t, H_t) \coloneqq \big(X_t, L_t(X_t)\big),$$

[6] "Conformal Invariance of Planar Loop-Erased Random Walks and Uniform Spanning Trees", *Annals of Probability*, 32(1B) (2004), pp. 939–95.

where for each time t, we look at both the position of the process X_t and the local time H_t spent on this point.

Heuristically, we say that $(X_t)_t$ is self-repelling when $(X_t)_t$ tends to avoid the most-visited points, that is, the points $x \in \mathbb{R}$ where $L_t(x)$ is large.

Figure 26 represents a snapshot at time t, with the position of the process (X_t, H_t), and its local time L_t.

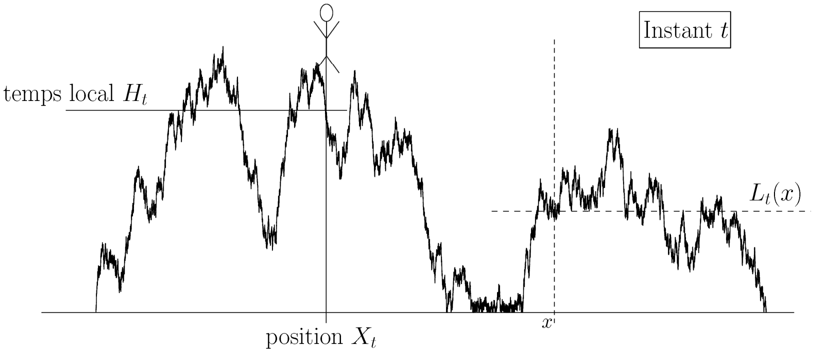


Figure 26: Process with its local time: snapshot of the moment t and its local time $L_t(\cdot)$

The self-repelling process has two remarkable properties. First, it is a recurrent process, in other words it visits each real point an infinite number of times. Second, it has a scaling property, like Brownian motion but with a different exponent:

For all $a > 0$, the processes $(X_{at}, t \geq 0)$ and $\left(a^{2/3} X_t, t \geq 0\right)$ have the same law (for Brownian motion, you will remember that the exponent was $a^{1/2}$).

4. Strategy of the fugitive

Now let us return to the initial question of flight along a straight line. How can one find a strategy to escape one's pursuers? The function $t \mapsto X_t$ represents the trajectory of the fugitive. The pursuers possess information about the number of times he has visited each point up to the present moment (thanks to hotel bills), but not the precise dates. So the pursuers know the local time, in other words the function $x \mapsto L(t, x)$ at time t. The aim is to find a continuous trajectory on the real number line that leaves the least possible information about the local time.

What happens if we follow a self-repelling process? In other words, what is the conditional law of X_1, the position of the self-repelling process at time 1 when we know $L_1(\cdot)$, that is, the time spent on each point?

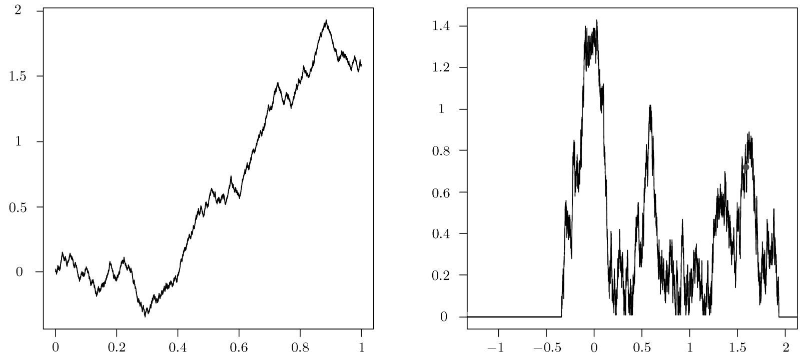


Figure 27: A clever (self-repelling) fugitive: on the left $(X_t, t \in [0,1])$, and on the right the local time $L_1(\cdot)$

In the above figure, the graph on the left represents a trajectory of the self-repelling process up until time 1. Imagine that we try to find out where the fugitive is at the moment 1. The available information can be found in the graph on the right: the curve corresponds to the **local time** at time 1, that is, the time that the fugitive has spent on each point before $t = 1$.

Looking at the local time, we can see that this process does leave a little bit of information: it is necessarily on a point of the support $[x_-, x_+]$ of $L_t(x)$ (where x_- is the left-most point visited and x_+ is the right-most point visited) and it is not completely hidden in the middle: there are points of strict growth (Figure 28) where the local time is equal to 0 and the trajectory only passes once.

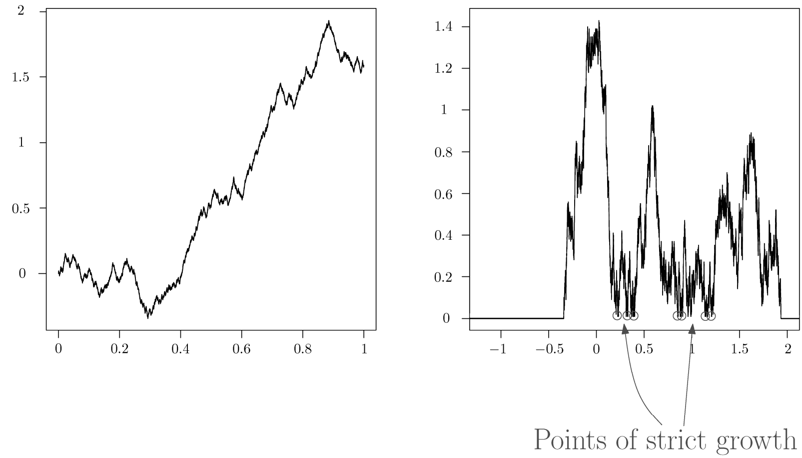


Figure 28: The trajectory on the left and the local time on the right with the points of strict growth

As we know that the process started at 0, we can deduce that the fugitive is necessarily in the interval between the last point of strict growth and the most-visited point, denoted by I (see Figure 29). Can we know more than this? The answer is no! The law of the point where X_1 is situated is **uniform over the interval I**. This is the subject of the theorem of Laure Dumaz (2013)[7]: the law of X_1 when we know $L_1(\cdot)$ is uniform over the interval I.[8]

So the fugitive is well hidden. Our intuition would probably tell us that the process is more likely to find itself in a frequently visited place, but in our case, the property of self-repellence counteracts this phenomenon.

[7] "Marginal Densities of the 'True' Self-Repelling Motion", with B. Tóth, in *Stochastic Processes and their Applications*, 123(4), April 2013, pp. 1454–71.

[8] This interval is the excursion (an excursion is a maximal interval over which the local time is non-zero) the furthest away from that which contains the starting point (the one furthest to the right in Figure 29).

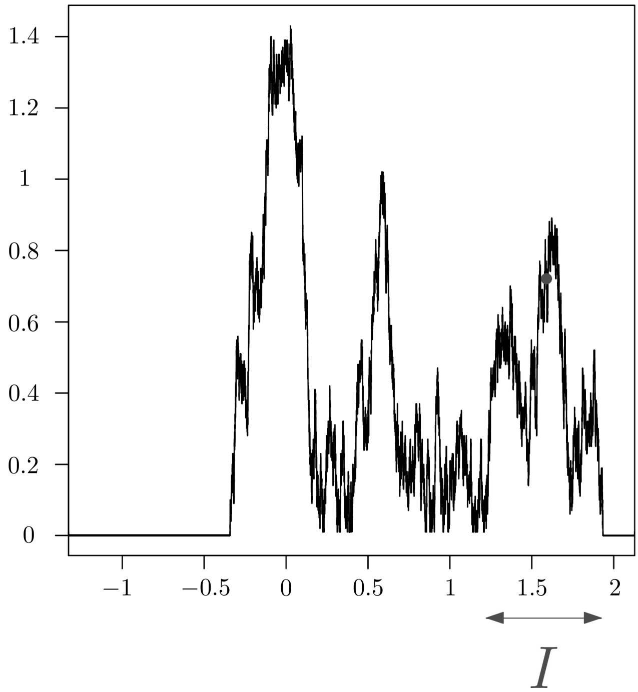


Figure 29: Definition of the interval I

Conclusion

This text concludes with some open questions. The first concerns the continuous process corresponding to the scaling limits of the self-repelling walk and of the toy model. It would be interesting to prove that it is universal, in the same way that Brownian motion is universal, in other words, that it corresponds to the scaling limit of a large class of discrete self-repelling models. Another question concerns the possible applications of the self-repelling process. Are there biological models corresponding to this process, or models in game theory for maximizing an evasion?