Some Topics in Stochastic Partial Differential Equations

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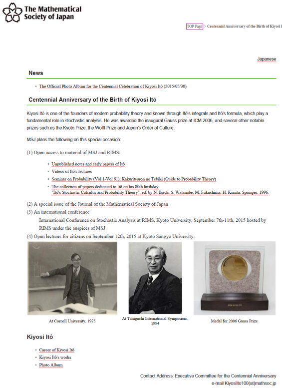
November 26, 2015

L' Héritage de Kiyosi Itô en perspective Franco-Japonaise, Ambassade de France au Japon

Plan of talk

1. Itô's SPDE
2. TDGL equation (Dynamic $P(\phi)$-model, Stochastic Allen-Cahn equation)
3. Kardar-Parisi-Zhang equation

Centennial Anniversary
of the Birth of Kiyosi Itô
by the Math. Soc. Japan



The Mathematical Society of Japan

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Japanese

News

- The Official Photo Album for the Centennial Celebration of Kiyosi Itô (2015/05/30)

Centennial Anniversary of the Birth of Kiyosi Itō

Kiyosi Itō is one of the founders of modern probability theory and known through Itō's integrals and Itō's formula, which play a fundamental role in stochastic analysis. He was awarded the inaugural Gauss prize at ICM 2006, and several other notable prizes such as the Kyoto Prize, the Wolff Prize and Japan's Order of Culture.

MSJ plans the following on this special occasion:

(1) Open access to material of MSJ and RIMS:

- Unpublished notes and early papers of Itô
- Videos of Itô's lectures
- Seminar on Probability (Vol 1-Vol 61), Kakuritsuron no Tebiki (Guide to Probability Theory)
- The collection of papers dedicated to Itô on his 80th birthday "Itô's Stochastic Calculus and Probability Theory", ed. by N. Ikeda, S. Watanabe, M. Fukushima, H. Kunita, Springer, 1996.

(2) A special issue of the Journal of the Mathematical Society of Japan

(3) An international conference

International Conference on Stochastic Analysis at RIMS, Kyoto University, September 7th-11th, 2015 hosted by RIMS under the auspices of MSJ

(4) Open lectures for citizens on September 12th, 2015 at Kyoto Sangyo University.

At Cornell University, 1975

At Taniguchi International Symposium, 1994

Medal for 2006 Gauss Prize

Kiyosi Itô

- Career of Kiyosi Itô
- Kiyosi Itô's works
- Photo Album

Contact Address: Executive Committee for the Centennial Anniversary
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1. Itô's SPDE

- Itô was interested in the following problem [2] (Math. Z. '83): Let $\{B_k(t)\}_{k=1}^{\infty}$ be independent 1D Brownian motions with common initial distribution μ. Set

$$u_n(t,dx) := \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}\delta_{B_k(t)}(dx) - E\left[\sum_{k=1}^{n}\delta_{B_k(t)}(dx)\right]\right).$$

- Then, $u_n(t,\cdot) \Rightarrow u(t,\cdot)dx$ and $u(t,\cdot)$ satisfies the SPDE:

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \partial_x\left(\sqrt{\mu(t,x)}\dot{W}(t,x)\right),$$

where $\dot{W}(t,x) = \dot{W}(t,x,\omega)$ is a space-time Gaussian white noise with covariance structure formally given by

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y), \tag{1}$$

and $\mu(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\mu(dy)$.

Proof is given as follows:

- For every test function $\varphi \in C_0^\infty(\mathbb{R})$,

$$u_n(t,\varphi) = \frac{1}{\sqrt{n}}\Big(\sum_{k=1}^n \varphi(B_k(t)) - E\Big[\sum_{k=1}^n \varphi(B_k(t))\Big]\Big).$$

- Applying Itô's formula, we have

$$du_n(t,\varphi) = \frac{1}{\sqrt{n}}\Big(\sum_{k=1}^n \partial_x\varphi(B_k(t))dB_k(t) + \frac{1}{2}\sum_{k=1}^n \partial_x^2\varphi(B_k(t))dt - \frac{1}{2}E[\cdots]dt\Big).$$

- drift term $= \frac{1}{2}u_n(t,\partial_x^2\varphi)dt$
- diffusion term $\frac{1}{\sqrt{n}}\sum_{k=1}^n \int_0^t \partial_x\varphi(B_k(s))dB_k(s)$ has a quadratic variation: $\frac{1}{n}\sum_{k=1}^n \int_0^t \partial_x\varphi(B_k(s))^2 ds$ which converges as $n\to\infty$ to $\int_0^t ds \int_{\mathbb{R}} \partial_x\varphi(x)^2\mu(s,x)dx$ by LLN.
- The limit $\int_0^t\int_{\mathbb{R}} \partial_x\varphi(x)\sqrt{\mu(s,x)}\dot{W}(s,x)dsdx$ has the same quad.var.

This result was extended by H. Spohn (CMP '86) to the interacting case under equilibrium: $dX_k(t) = -\frac{1}{2}\sum_{i\neq k}\nabla V(X_k(t)-X_i(t))dt + dB_k(t)$.

2. TDGL equation

- Time-dependent Ginzburg-Landau (TDGL) equation (cf. Hohenberg-Halperin, Kawasaki-Ohta, Langevin equation)

$$\partial_t u = -\frac{1}{2}\frac{\delta H}{\delta u(x)}(u) + \dot{W}(t,x), \ x \in \mathbb{R}^d,$$

$\dot{W}(t,x)$: space-time Gaussian white noise

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}|\nabla u(x)|^2 + V(u(x)) \right\} dx.$$

- Heuristically, Gibbs measure $\frac{1}{Z}e^{-H}du$ is invariant under these dynamics, where $du = \prod_{x\in\mathbb{R}^d} du(x)$.

- Since the functional derivative is given by

$$\frac{\delta H}{\delta u(x)} = -\Delta u + V'(u(x)),$$

TDGL eq has the form:

$$\partial_t u = \frac{1}{2}\Delta u - \frac{1}{2}V'(u) + \dot{W}(t,x). \tag{2}$$

- The noise $\dot{W}(t,x)$ can be constructed as follows:
  Take $\{\psi_k\}_{k=1}^{\infty}$: CONS of $L^2(\mathbb{R}^d, dx)$ and $\{B_k(t)\}_{k=1}^{\infty}$: independent 1D BMs, and consider a (formal) Fourier series:

$$W(t,x) = \sum_{k=1}^{\infty} B_k(t)\psi_k(x). \tag{3}$$

Stochastic PDEs used in physics are sometimes ill-posed.

For TDGL eq (2),

- Noise is very irregular: $\dot{W} \in C^{-\frac{d+1}{2}-} := \cap_{\delta>0} C^{-\frac{d+1}{2}-\delta}$ a.s.
- Linear case (without $V'(u)$): $u(t,x) \in C^{\frac{2-d}{4}-,\frac{2-d}{2}-}$ a.s.
- Well-posed only when $d=1$.

Martin Hairer:

- Theory of regularity structures, systematic renormalization
- TDGL equation with $V(u) = \frac{1}{4}u^4$:
  =Stochastic quantization (Dynamic $P(\phi)_d$-model):

$$\partial_t \phi = \Delta \phi - \phi^3 + \dot{W}(t,x), \quad x \in \mathbb{R}^d$$

- For $d = 2$ or 3, replace $\dot{W}$ by a smeared noise $\dot{W}^\varepsilon$ and introduce a renormalization factor $-C_\varepsilon \phi$. Then, the limit of $\phi = \phi^\varepsilon$ as $\varepsilon \downarrow 0$ exists (locally in time).
- The solution is continuous in $\dot{W}^\varepsilon$ and their (finitely many) polynomials.

Another approaches

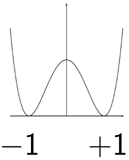
- Gubinelli and others:
  Paracontrolled distributions (harmonic analytic method)
- Kupiainen

- When $\dot{W} = 0$ (noise is not added) and $V=$ double-well type, TDGL eq (2) is known as Allen-Cahn equation or reaction-diffusion equation of bistable type.
- Dynamic phase transition, Sharp interface limit as $\varepsilon \downarrow 0$ for TDGL equation (=stochastic Allen-Cahn equation):

$$\partial_t u = \Delta u + \frac{1}{\varepsilon} f(u) + \dot{W}(t,x), \quad x \in \mathbb{R}^d \tag{4}$$

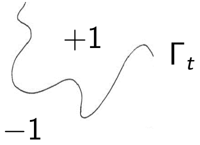
$f = -V'$, Potential V is of double-well type:

e.g., $f = u - u^3$ if $V = \frac{1}{4}u^4 - \frac{1}{2}u^2$



- The limit is expected to satisfy:

$$u(t,x) \xrightarrow[\varepsilon\downarrow 0]{} \begin{cases} +1 \\ -1 \end{cases}$$

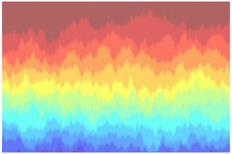


- A random phase separating hyperplane Γ_t appears and its time evolution is studied under proper time scaling.

3. Kardar-Parisi-Zhang equation

- The KPZ (Kardar-Parisi-Zhang, 1986) equation describes the motion of growing interface with random fluctuation.
- It has the form for height function $h(t,x)$:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}. \qquad (5)$$



Ill-posedness of KPZ eq (5):

- The nonlinearity and roughness of the noise do not match.
- The linear SPDE:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \dot{W}(t,x),$$

obtained by dropping the nonlinear term has a solution $h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty)\times\mathbb{R})$ a.s. Therefore, no way to define the nonlinear term $(\partial_x h)^2$ in (5) in a usual sense.
- Actually, the following Renormalized KPZ eq with compensator $\delta_x(x)$ $(= +\infty)$ has the meaning:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x),$$

as we will see later.

- $\frac{1}{3}$-**power law**: Under stationary situation,

$$\mathrm{Var}(h(t,0)) = O(t^{\frac{2}{3}})$$

  as $t \to \infty$, i.e. the fluctuations of $h(t,0)$ are of order $t^{\frac{1}{3}}$. Subdiffusive behavior different from CLT (=diffusive behavior).
- (Sasamoto-Spohn) The limit distribution of $h(t,0)$ under scaling is given by the so-called Tracy-Widom distribution (different depending on initial distributions).

Cole-Hopf solution to the KPZ equation

- Consider the linear stochastic heat equation (SHE) for $Z = Z(t,x)$:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x), \qquad (6)$$

  with a multiplicative noise. This eq is well-posed (if we understand the multiplicative term in Itô's sense but ill-posed in Stratonovich's sense).
- If $Z(0,\cdot) > 0 \Rightarrow Z(t,\cdot) > 0$.
- Therefore, we can define the Cole-Hopf transformation:

$$h(t,x) := \log Z(t,x). \qquad (7)$$

  This is called Cole-Hopf solution of KPZ equation.

Heuristic derivation of the KPZ eq (with renormalization factor $\delta_x(x)$) from SHE (6) under the Cole-Hopf transformation (7):

- Apply Itô's formula for $h = \log z$:

$$\begin{aligned}
\partial_t h &= Z^{-1}\partial_t Z - \tfrac{1}{2} Z^{-2}(\partial_t Z)^2 \\
&= Z^{-1}\left(\tfrac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \tfrac{1}{2}\delta_x(x) \\
&\qquad\qquad \text{by SHE (6) and } (dZ(t,x))^2 = (ZdW(t,x))^2 \\
&\qquad\qquad\qquad\qquad dW(t,x)dW(t,y) = \delta(x-y)dt \\
&= \tfrac{1}{2}\{\partial_x^2 h + (\partial_x h)^2\} + \dot{W} - \tfrac{1}{2}\delta_x(x)
\end{aligned}$$

- This leads to the Renormalized KPZ eq:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x). \qquad (8)$$

- The Cole-Hopf solution $h(t,x)$ defined by (7) is meaningful, although the equation (5) does not make sense.

- Goal is to introduce approximations for (8).
- Hairer (2013, 2014) gave a meaning to (8) without bypassing SHE.

KPZ approximating equation-1: Simple

- Symmetric convolution kernel Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x)dx = 1$ be given, and set $\eta^\varepsilon(x) := \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ for $\varepsilon > 0$.
- Smeared noise The smeared noise is defined by

$$W^\varepsilon(t,x) = \langle W(t), \eta^\varepsilon(x - \cdot)\rangle \, \big(= W(t) * \eta^\varepsilon(x)\big).$$

- Approximating Eq-1:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\big((\partial_x h)^2 - \xi^\varepsilon\big) + \dot{W}^\varepsilon(t,x)$$
$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\dot{W}^\varepsilon(t,x),$$

  where $\xi^\varepsilon = \eta_2^\varepsilon(0)\ (:= \eta^\varepsilon * \eta^\varepsilon(0))$.
- It is easy to show that $Z = Z^\varepsilon$ converges to the sol Z of (SHE), and therefore $h = h^\varepsilon$ converges to the Cole-Hopf solution of the KPZ eq.

KPZ approximating equation-2 (jointly with Quastel):

- We want to introduce another approximation which is suitable to study the invariant measures.
- **General principle.** Consider the SPDE

$$\partial_t h = F(h) + \dot{W},$$

and let A be a certain operator. Then, the structure of the invariant measures essentially does not change for

$$\partial_t h = A^2 F(h) + A\dot{W}.$$

- This leads to

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x), \qquad (9)$$

where $\eta_2(x) = \eta * \eta(x)$, $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$ and $\xi^\varepsilon = \eta_2^\varepsilon(0)$.

Cole-Hopf transform for SPDE (9)

- The goal is to pass to the limit $\varepsilon \downarrow 0$ in the KPZ approximating equation (9):

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x).$$

- We consider its Cole-Hopf transform: $Z\,(\equiv Z^\varepsilon) := e^h$. Then, by Itô's formula, Z satisfies the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + A^\varepsilon(x,Z) + Z\dot{W}^\varepsilon(t,x), \tag{10}$$

where

$$A^\varepsilon(x,Z) = \frac{1}{2}Z(x)\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Z}{Z}\right)^2(x)\right\}.$$

- The complex term $A^\varepsilon(x,Z)$ looks vanishing as $\varepsilon \downarrow 0$.

- But this is not true. Indeed, under the average in time t, $A^\varepsilon(x, Z)$ can be replaced by a linear function $\frac{1}{24} Z$.
- The limit as $\varepsilon \downarrow 0$ (under stationarity of tilt),

$$\partial_t Z = \frac{1}{2} \partial_x^2 Z + \frac{1}{24} Z + Z \dot{W}(t, x).$$

- Or, heuristically at KPZ level,

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} \{(\partial_x h)^2 - \delta_x(x)\} + \frac{1}{24} + \dot{W}(t, x).$$

Multi-component KPZ equation can be also discussed:

- Ferrari-Sasamoto-Spohn (2013) studied $\mathbb{R}^d$-valued KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{R}$:

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha(t,x), \; x \in \mathbb{R}, \quad (11)$$

  where $\dot{W}(t,x) = (\dot{W}^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise. The constants $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ satisfy the condition:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\gamma_{\beta\alpha}. \quad (12)$$

- Similar SPDE appears to discuss motion of loops on a manifold, cf. Funaki (1992).

Summary of talk

1. Itô's SPDE
2. TDGL equation
   (Dynamic $P(\phi)$-model, Stochastic Allen-Cahn equation)
3. KPZ equation

Thank you for your attention!